Answer Key for the Practice Problems.

1 For each of the following vectors $\mathbf{x}$, find the projection of $\mathbf{x}$ in the direction of $\mathbf{v}$.

(a) Find the projection of $\mathbf{x} = (3, 3, 3)$ in the direction of $\mathbf{v} = (2, 4, 6)$.

(b) Find the projection of $\mathbf{x} = (1, 2, 3)$ in the direction of $\mathbf{v} = (1, 1, 1)$.

(c) Find the projection of $\mathbf{x} = (1, 2)$ in the direction of $\mathbf{v} = (3, 4)$.

(a) $\text{proj}_v x = \frac{\langle x,v\rangle}{\langle v,v\rangle} v$; $\langle x,v\rangle = \begin{pmatrix}3\\3\\3\end{pmatrix}\cdot\begin{pmatrix}2\\4\\6\end{pmatrix} = 6+12+18 = 36$;

$\langle v,v\rangle = \begin{pmatrix}2\\4\\6\end{pmatrix}\cdot\begin{pmatrix}2\\4\\6\end{pmatrix} = 4+16+36 = 56$; $\text{proj}_v x = \frac{36}{56} v = \boxed{\frac{9}{14}\begin{pmatrix}2\\4\\6\end{pmatrix}}$;

(b) $\text{proj}_v x = \frac{\langle x,v\rangle}{\langle v,v\rangle} v$; $\langle x,v\rangle = \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 1+2+3 = 6$;

$\langle v,v\rangle = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 1+1+1 = 3$; $\text{proj}_v x = \frac{6}{3} v = \boxed{2\begin{pmatrix}1\\1\\1\end{pmatrix}}$;

(c) $\text{proj}_v x = \frac{\langle x,v\rangle}{\langle v,v\rangle} v$; $\langle x,v\rangle = \begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} = 3+8 = 11$;

$\langle v,v\rangle = \begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix} = 9+16 = 25$; $\text{proj}_v x = \boxed{\frac{11}{25}\begin{pmatrix}3\\4\end{pmatrix}}$;

2 For each of the following lines, provide a **parametrization** of the line in the form $\mathbf{r}(t) = \mathbf{p_0} + t\mathbf{d}$ for $\mathbf{p_0}, \mathbf{d} \in \mathbb{R}^n$ for appropriate n.

(a) The line in $\mathbb{R}^3$ passing through $\mathbf{p} = (1, 2, 3)$ with direction vector $\mathbf{d} = (4, 5, 6)$.

(b) The line in $\mathbb{R}^2$ satisfying the equation $y = \frac{3}{2}x + 4$

(c) The line in $\mathbb{R}^3$ passing through the points $\mathbf{p} = (3, 2, 1)$ and $\mathbf{q} = (1, 2, 3)$.

(a) $r(t) = p_0 + td = p + td = \boxed{\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}4\\5\\6\end{pmatrix}}$;

(b) With $y = \frac{r}{q}x + b$: $d = \begin{pmatrix}q\\r\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$; $p_0 = \begin{pmatrix}0\\b\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}$; $\boxed{r(t) = \begin{pmatrix}0\\4\end{pmatrix} + t\begin{pmatrix}2\\3\end{pmatrix}}$

(c) $d = q - p = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\2\end{pmatrix}$; $r(t) = p + td = \boxed{\begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}-2\\0\\2\end{pmatrix}}$;

3 For each of the following planes, determine the **implicit equation** of the plane in the form $ax+by+cz = d$ for scalars $a, b, c, d \in \mathbb{R}$.

(a) The plane passing through the points $\mathbf{p_1} = (3, 4, 5)$, $\mathbf{p_2} = (2, 3, 1)$, and $\mathbf{p_3} = (5, 5, 2)$.

(b) The plane with normal vector $\mathbf{N} = (1, 2, 3)$ passing through the point $\mathbf{p} = (3, 2, 1)$.

(c) The plane with parametrization $P(t, s) = (1, 2, 3) + (1, 2, 0)t + (0, 2, 1)s$ for $t, s \in \mathbb{R}$.

(d) The plane with direction vectors $\mathbf{d_1} = (1, 2, 2)$ and $\mathbf{d_2} = (2, 2, 1)$ intersecting the origin.

(a) $d_1 = p_1 - p_2 = \begin{pmatrix}3\\4\\5\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}1\\1\\4\end{pmatrix}$; $d_2 = p_3 - p_2 = \begin{pmatrix}5\\5\\2\end{pmatrix} - \begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$;

$N = d_1 \times d_2 = \begin{pmatrix}1\\1\\4\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix} = \det\begin{pmatrix}i & j & k\\1 & 1 & 4\\3 & 2 & 1\end{pmatrix} = (i)[(1)(1)-(4)(2)] + (-j)[(1)(1)-(4)(3)]$

$+ k[(1)(2)-(3)(1)] = (i)(-7) + (j)(11) + (k)(-1) = \begin{pmatrix}-7\\11\\-1\end{pmatrix}$;

implicit equation: $\langle N, \begin{pmatrix}x\\y\\z\end{pmatrix}\rangle = \langle N, p_0\rangle$; $\langle N, \begin{pmatrix}x\\y\\z\end{pmatrix}\rangle = \begin{pmatrix}-7\\11\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = -7x + 11y - z$;

$\langle N, p_0\rangle = \langle N, p_2\rangle = \begin{pmatrix}-7\\11\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix} = -14 + 33 - 1 = 18$; Ans: $\boxed{-7x + 11y - z = 18}$;

(b) $\langle N, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 2y + 3z$; $\langle N, p_0 \rangle = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3 + 4 + 3 = 10$;

implicit eqn: $\boxed{x + 2y + 3z = 10}$;

(c) $d_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$; $d_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$; $N = d_1 \times d_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \det \begin{pmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} = i[(2)(1) - \cancel{(0)(2)}]$

$+ (-j)[(1)(1) - \cancel{(0)(0)}] + k[(1)(2) - \cancel{(2)(0)}] = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$;

$\langle N, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rangle = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2x - y + 2z$; $p_0 = P(0,0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $\langle N, p_0 \rangle = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$= 2 + (-2) + (6) = 6$; Answer: $\boxed{2x - y + 2z = 6}$;

(d) $N = d_1 \times d_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \det \begin{pmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix} = i[(2)(1) - (2)(2)] + (-j)[(1)(1) - (2)(2)]$

$+ k[(1)(2) - (2)(2)] = i(2-4) + j(-1+4) + k(2-4) = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$; $\langle N, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rangle = -2x + 3y - 2z$;

passing through the origin: $p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$; $\langle N, p_0 \rangle = \begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$; Answer: $\boxed{-2x + 3y - 2z = 0}$;

4 For each matrix, determine if the matrix is invertible. If it is invertible, find its inverse.

(a) $\mathbf{A_1} = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$

(b) $\mathbf{A_2} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

(c) $\mathbf{A_3} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

(d) $\mathbf{A_4} = \begin{pmatrix} 1 & 3 & -5 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$

(e) $\mathbf{A_5} = \begin{pmatrix} 2 & 0 & 4 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{pmatrix}$

(f) $\mathbf{A_6} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

(a) $\det(A_1) = (1)(-6) - (3)(-2) = -6 + 6 = 0$; A_1 is $\boxed{\text{not invertible}}$

(b) Using the formula: $\det(A_2) = (3)(3) - (2)(2) = 9 - 4 = 5$; $\boxed{A_2^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}}$;

(c) Method 1: Gaussian Elimination on $(A\ I_3)$ to get $(I_3\ A^{-1})$;

$$(A_3\ I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-3R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right) \xrightarrow{-4R_2 + R_3 \mapsto R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 12 & 5 & -4 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{12}R_3 \mapsto R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{array}\right) \xrightarrow[5R_3 + R_2 \mapsto R_2]{-3R_3 + R_1 \mapsto R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -1 & 0 & \frac{1}{12} & -\frac{2}{3} & \frac{5}{12} \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{array}\right) \xrightarrow{R_1 + 2R_2 \mapsto R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{12} & -\frac{1}{3} & \frac{7}{12} \\ 0 & -1 & 0 & \frac{1}{12} & -\frac{2}{3} & \frac{5}{12} \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{array}\right) \xrightarrow{(-1)R_2 \mapsto R_2} \left(I_3 \;\; \begin{array}{ccc} -\frac{1}{12} & -\frac{1}{3} & \frac{7}{12} \\ -\frac{1}{12} & \frac{2}{3} & -\frac{5}{12} \\ \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{array} \right);$$

$$\boxed{A_3^{-1} = \begin{pmatrix} -\frac{1}{12} & -\frac{1}{3} & \frac{7}{12} \\ -\frac{1}{12} & \frac{2}{3} & -\frac{5}{12} \\ \frac{5}{12} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} -1 & -4 & 7 \\ -1 & 8 & -5 \\ 5 & -4 & 1 \end{pmatrix}};$$

Method 2: Cofactor Matrix. See LN4B. This method will not be on the final exam but you may use it.

$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$; Calculating the cofactors:

First Row: $C[1,1] = (-1)^{1+1}\det\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = (3-2) = 1$; $C[1,2] = (-1)^{1+2}\det\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = (-1)(2-3) = 1$;

$C[1,3] = (-1)^{1+3}\det\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = (4-9) = -5$;

Determinant: $\det(A_3) = (1)C[1,1] + (2)C[1,2] + (3)C[1,3] = (1)(1) + (2)(1) + (3)(-5) = 1+2-15 = -12$;

Since $\det(A_3) \neq 0$, A_3 is invertible.

Second Row: $C[2,1] = (-1)^{2+1}\det\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = (-1)(2-6) = 4$; $C[2,2] = (-1)^{2+2}\det\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} = (1-9) = -8$;

$C[2,3] = (-1)^{2+3}\det\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = (-1)(2-6) = 4$;

Third Row: $C[3,1] = (-1)^{3+1}\det\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} = (2-9) = -7$; $C[3,2] = (-1)^{3+2}\det\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = (-1)(1-6) = 5$;

$C[3,3] = (-1)^{3+3}\det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = (3-4) = -1$;

Cofactor Matrix: $C = \begin{pmatrix} 1 & 1 & -5 \\ 4 & -8 & 4 \\ -7 & 5 & -1 \end{pmatrix}$; Inverse: $A_3^{-1} = \frac{1}{\det(A_3)} C^T = \boxed{-\frac{1}{12}\begin{pmatrix} 1 & 4 & -7 \\ 1 & -8 & 5 \\ -5 & 4 & -1 \end{pmatrix}}$;

(d) $A_4 = \begin{pmatrix} 1 & 3 & -5 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$ is $\boxed{\text{not invertible}}$.

Method 1: Determine linear dependence by guessing. Useful if you can see it immediately.

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-2)\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$; The columns of A_3 are lin. dependent $\Rightarrow$ A_3 is not invertible.

★**Method 2:** Determine linear dependence by Gaussian Elimination.

$$A_4 = \begin{pmatrix} 1 & 3 & -5 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-2R_1+R_2 \mapsto R_2} \begin{pmatrix} 1 & 3 & -5 \\ 0 & -4 & 8 \\ 0 & -8 & 16 \end{pmatrix} \xrightarrow{-2R_2+R_3 \mapsto R_3} \begin{pmatrix} 1 & 3 & -5 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{pmatrix};$$

The 3rd column does NOT have a pivot.

$\therefore$ A_4 has linearly dependent columns; $\therefore$ A_4 is not invertible.

Note: Doing Gaussian Elimination on $(A_4\ I_3)$ will yield the same 3 columns.

Since at least one of the first 3 columns does not have a pivot, A_4 is not invertible.

Method 3: Determine invertibility by the determinant.

$$\det(A_3) = \det\begin{pmatrix} 1 & 3 & -5 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix} = (1)\det\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} + (-1)(2)\det\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix} + (3)\det\begin{pmatrix} 3 & -5 \\ 2 & -2 \end{pmatrix}$$

$$= (1)[2-(-2)] + (-2)[3-(-5)] + (3)[-6-(-10)] = (1)(4) + (-2)(8) + (3)(4)$$

$$= 4 - 16 + 12 = 0;$$

Since $\det(A_3) = 0$, A_3 is not invertible.

(e) **Method 1.** We can use Gaussian Elimination on $(A_5\ I_3)$ to get $(I_3\ A_5^{-1})$;

$$(A_5\ I_3) = \begin{pmatrix} 2 & 0 & 4 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_1+R_2 \mapsto R_2} \begin{pmatrix} 2 & 0 & 4 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{-R_2+R_3 \mapsto R_3} \begin{pmatrix} 2 & 0 & 4 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 0 & 8 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[2R_2 \mapsto R_2]{2R_1 \mapsto R_1} \begin{pmatrix} 4 & 0 & 8 & 2 & 0 & 0 \\ 0 & 6 & -8 & -2 & 2 & 0 \\ 0 & 0 & 8 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[R_2+R_3 \mapsto R_2]{R_1+(-1)R_3 \mapsto R_1} \begin{pmatrix} 4 & 0 & 0 & 1 & 1 & -1 \\ 0 & 6 & 0 & -1 & 1 & 1 \\ 0 & 0 & 8 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[\substack{\frac{1}{6}R_2 \mapsto R_2 \\ \frac{1}{8}R_3 \mapsto R_3}]{\frac{1}{4}R_1 \mapsto R_1} \left(I_3 \;\; \begin{matrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{matrix} \right);$$

$$A_5^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \end{pmatrix} = \boxed{\frac{1}{24}\begin{pmatrix} 6 & 6 & -6 \\ -4 & 4 & 4 \\ 3 & -3 & 3 \end{pmatrix}};$$

Method 2. Branching from Method 1.

$$(A_5\ I_3) \to \cdots \to \begin{pmatrix} 4&0&0&1&1&-1 \\ 0&6&0&-1&1&1 \\ 0&0&8&1&-1&1 \end{pmatrix} \xrightarrow[\substack{4R_2 \mapsto R_2 \\ 3R_3 \mapsto R_3}]{6R_1 \mapsto R_1} \begin{pmatrix} 24&0&0&6&6&-6 \\ 0&24&0&-4&4&4 \\ 0&0&24&3&-3&3 \end{pmatrix} = \left(24I_3\ \begin{matrix} 6&6&-6 \\ -4&4&4 \\ 3&-3&3 \end{matrix}\right);$$

$$\boxed{A_5^{-1} = \frac{1}{24}\begin{pmatrix} 6&6&-6 \\ -4&4&4 \\ 3&-3&3 \end{pmatrix}};$$

Algebraically: let $B = \begin{pmatrix} 6&6&-6 \\ -4&4&4 \\ 3&-3&3 \end{pmatrix}$; $A_5(A_5^{-1}) = I_3$ yields $24I_3(A_5^{-1}) = B$; Then, $A_5^{-1} = \frac{1}{24}B$;

(f) $$(A_6\ I_3) = \begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&1&0&1&0 \\ 1&0&1&0&0&1 \end{pmatrix} \xrightarrow{-R_1+R_3 \mapsto R_3} \begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&1&0&1&0 \\ 0&-1&1&-1&0&1 \end{pmatrix} \xrightarrow{R_2+R_3 \mapsto R_3} \begin{pmatrix} 1&1&0&1&0&0 \\ 0&1&1&0&1&0 \\ 0&0&2&-1&1&1 \end{pmatrix} \xrightarrow[2R_2 \mapsto R_2]{2R_1 \mapsto R_1}$$

$$\begin{pmatrix} 2&2&0&2&0&0 \\ 0&2&2&0&2&0 \\ 0&0&2&-1&1&1 \end{pmatrix} \xrightarrow{R_2+(-1)R_3 \mapsto R_2} \begin{pmatrix} 2&2&0&2&0&0 \\ 0&2&0&1&1&-1 \\ 0&0&2&-1&1&1 \end{pmatrix} \xrightarrow{R_1+(-1)R_2 \mapsto R_1} \begin{pmatrix} 2&0&0&1&-1&1 \\ 0&2&0&1&1&-1 \\ 0&0&2&-1&1&1 \end{pmatrix} = (2I_3\ A_6^{-1});$$

$$\boxed{A_6^{-1} = \frac{1}{2}\begin{pmatrix} 1&-1&1 \\ 1&1&-1 \\ -1&1&1 \end{pmatrix}};$$

5. Determine the solution set of the following systems, i.e. identify **all** solutions. If there are infinite solutions, express them as a linear combination of vectors with free variables as the scalars.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

(f) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & -4 \\ 1 & -2 & 3 & -4 \\ 1 & 2 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 3 & 4 \\ 1 & 2 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$

(a) $$\begin{pmatrix} 1&2 \\ 0&3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}; \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1&2 \\ 0&3 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{3-0}\begin{pmatrix} 3&-2 \\ -0&1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 12-10 \\ 5 \end{pmatrix} = \boxed{\frac{1}{3}\begin{pmatrix} 2 \\ 5 \end{pmatrix}}$$

(b) $\begin{pmatrix} 2&-4 \\ -3&6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; Remark: $\begin{pmatrix} 2&-4 \\ -3&6 \end{pmatrix}$ is NOT invertible.

Aug. Mat: $$\begin{pmatrix} 2&-4&1 \\ -3&6&2 \end{pmatrix} \xrightarrow[2R_2 \mapsto R_2]{3R_1 \mapsto R_1} \begin{pmatrix} 6&-12&3 \\ -6&12&4 \end{pmatrix} \xrightarrow{R_1+R_2 \mapsto R_2} \begin{pmatrix} 6&-12&3 \\ 0&0&7 \end{pmatrix};$$ Last column has a pivot. $\therefore$ $\boxed{\text{NO SOLUTIONS.}}$

(c) $\begin{pmatrix} 1&-3 \\ 2&-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$; Remark: $\begin{pmatrix} 1&-3 \\ 2&-6 \end{pmatrix}$ is NOT invertible.

Aug. Mat: $$\begin{pmatrix} 1&-3&7 \\ 2&-6&14 \end{pmatrix} \xrightarrow{-2R_1+R_2 \mapsto R_2} \begin{array}{c} \begin{matrix} x_1 & x_2 & \end{matrix} \\ \begin{pmatrix} 1&-3&7 \\ 0&0&0 \end{pmatrix} \end{array};$$ Infinitely many solutions since x_2 is a free variable!

Back-Sub: $x_1 - 3x_2 = 7$; $x_1 = -3x_2 + 7$;

Solutions: $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2+7 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ 0 \end{pmatrix};$$ Ans: $\boxed{\left\{ x_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R} \right\}};$

(d) $\begin{pmatrix} 1&2&3 \\ 2&5&8 \\ 3&4&5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$;

Aug. Mat: $$\begin{pmatrix} 1&2&3&4 \\ 2&5&8&4 \\ 3&4&5&4 \end{pmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-2R_1+R_2 \mapsto R_2} \begin{pmatrix} 1&2&3&4 \\ 0&1&2&-4 \\ 0&-2&-4&8 \end{pmatrix} \xrightarrow{2R_2+R_3 \mapsto R_3} \begin{pmatrix} 1&2&3&4 \\ 0&1&2&-4 \\ 0&0&0&-16 \end{pmatrix};$$ $\boxed{\text{No Solutions.}}$

(e) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 4 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$;

Aug. Mat: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 5 \\ 3 & 4 & 9 & 6 \end{pmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-2R_1+R_2 \mapsto R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & -2 & 0 & -6 \end{pmatrix} \xrightarrow{2R_2+R_3 \mapsto R_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & -12 \end{pmatrix} \xrightarrow{\frac{1}{4}R_3 \mapsto R_3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -3 \end{pmatrix}$;

Option 1: Back-Sub Now.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= -3 \\ x_3 &= -3 \end{aligned}$$
;
$x_2 = -3 - 2x_3 = -3 - 2(-3) = -3 + 6 = 3$; $x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(3) - 3(-3) = 4 - 6 + 9 = 7$; Ans: $\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix}}$;

Option 2. Proceed to RREF, then Back-Sub.

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -3 \end{pmatrix} \xrightarrow[R_2-2R_3 \mapsto R_2]{R_1-3R_3 \mapsto R_1} \begin{pmatrix} 1 & 2 & 0 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_1-2R_2 \mapsto R_1} \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{pmatrix}$; Ans: $\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix}}$;

(f) $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;

Aug. Mat: $\begin{pmatrix} 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 0 \\ 6 & 7 & 8 & 0 \end{pmatrix} \xrightarrow{2R_2 \mapsto R_2} \begin{pmatrix} 2 & 3 & 4 & 0 \\ 6 & 8 & 10 & 0 \\ 6 & 7 & 8 & 0 \end{pmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-3R_1+R_2 \mapsto R_2} \begin{pmatrix} 2 & 3 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{pmatrix} \xrightarrow{-2R_2+R_3 \mapsto R_3} \begin{pmatrix} 2 & 3 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{R_1+3R_2 \mapsto R_1} \begin{pmatrix} 2 & 0 & -2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[-R_2 \mapsto R_2]{\frac{1}{2}R_1 \mapsto R_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; x_3 is a free variable.

Back-Sub: $\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$; $\begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \end{aligned}$; $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$; Solution: $\boxed{\left\{ x_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}}$;

(g) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -3 & -4 \\ 1 & -2 & 3 & -4 \\ 1 & 2 & 3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \end{pmatrix}$;

Aug. Mat: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 2 & -3 & -4 & 5 \\ 1 & -2 & 3 & -4 & 5 \\ 1 & 2 & -3 & -4 & 5 \end{pmatrix} \xrightarrow[\substack{-R_1+R_3 \mapsto R_3 \\ -R_1+R_4 \mapsto R_4}]{R_1+R_2 \mapsto R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 0 & 10 \\ 0 & -4 & 0 & -8 & 0 \\ 0 & 0 & -6 & -8 & 0 \end{pmatrix} \xrightarrow{R_2+R_3 \mapsto R_3} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 0 & 10 \\ 0 & 0 & 0 & -8 & 10 \\ 0 & 0 & 6 & 8 & 0 \end{pmatrix} \xrightarrow{R_3+R_4 \mapsto R_4}$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 0 & 10 \\ 0 & 0 & 0 & -8 & 10 \\ 0 & 0 & 6 & 0 & 10 \end{pmatrix}$; Back-Sub: $\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ 4x_2 &= 10; \\ 6x_3 &= 10; \\ -8x_4 &= 10; \end{aligned}$; $x_2 = \frac{5}{2}$; $x_3 = \frac{5}{3}$; $x_4 = -\frac{5}{4}$;

$$\begin{aligned} x_1 &= -2x_2 - 3x_3 - 4x_4 + 5 \\ &= -2\left(\tfrac{5}{2}\right) - 3\left(\tfrac{5}{3}\right) - 4\left(-\tfrac{5}{4}\right) + 5 \\ &= -5 - 5 + 5 + 5 = 0; \end{aligned}$$
; $\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{5}{2} \\ \frac{5}{3} \\ -\frac{5}{4} \end{pmatrix}}$;

(h) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & 3 & 4 \\ 1 & 2 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$;

Aug. Mat: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & -2 & 3 & 4 & -5 \\ 1 & 2 & -3 & -4 & 5 \end{pmatrix} \xrightarrow[-R_1+R_3 \mapsto R_3]{R_1+R_2 \mapsto R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 8 & 0 \\ 0 & 0 & -6 & -8 & 0 \end{pmatrix} \xrightarrow{R_2+R_3 \mapsto R_3} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 6 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2 \mapsto R_2}$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-R_2 \mapsto R_1} \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$; x_2 and x_4 are free variables.

Back-Sub: $\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_3 + 4x_4 &= 0 \end{aligned}$; $\begin{aligned} x_1 &= -2x_2 + 5 \\ x_3 &= -\tfrac{4}{3}x_4 \end{aligned}$;

Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2+5 \\ x_2 \\ -\frac{4}{3}x_4 \\ x_4 \end{pmatrix} \boxed{= x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for all } x_2, x_4 \in \mathbb{R}.}$

⑥ Determine the **left multiplication matrices** for the following linear transformations.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $T(\mathbf{e}_2) = \begin{pmatrix}3\\4\\5\end{pmatrix}$, and $T(\mathbf{e}_1) = \begin{pmatrix}7\\8\\9\end{pmatrix}$

(b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+2x_2+3x_3\\4x_2+5x_3\\6x_1+7x_3\end{pmatrix}$

(c) The reflection $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ across the line $y = \frac{1}{2}x$.

(d) The rotation $T_4 : \mathbb{R}^2 \to \mathbb{R}^2$ by $\theta = \frac{5\pi}{6}$ counterclockwise.

(e) The orthogonal projection $T_5 : \mathbb{R}^2 \to \mathbb{R}^2$ by onto the line spanned by $\mathbf{v} = \begin{pmatrix}3\\-5\end{pmatrix}$

(f) The reflection transformation $T_6 : \mathbb{R}^3 \to \mathbb{R}^3$ across the plane given by $x + 2x - 3z = 0$.

(g) The orthogonal projection $T_7 : \mathbb{R}^3 \to \mathbb{R}^3$ onto the plane with normal vector $\mathbf{N} = (1, -2, 3)^\top$ passing through the origin.

(a) $A_1 = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix}1&3&7\\2&4&8\\3&5&9\end{pmatrix}}$;

(b) $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\6\end{pmatrix}$; $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\4\\0\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\5\\7\end{pmatrix}$; $\boxed{A_2 = \begin{pmatrix}1&2&3\\0&4&5\\6&0&7\end{pmatrix}}$;

(c) The line $y = \frac{1}{2}x$ is spanned by $v = \begin{pmatrix}2\\1\end{pmatrix}$; let $w = \begin{pmatrix}-1\\2\end{pmatrix}$. Observe that $v \perp w$ and $\{v, w\}$ is a basis of $\mathbb{R}^2$;
Since T_3 is a reflection: $T(v) = v$ and $T(w) = -w$;
$\therefore A_3\begin{pmatrix}2&-1\\1&2\end{pmatrix} = \begin{pmatrix}2&1\\1&-2\end{pmatrix}$; $A_3 = \begin{pmatrix}2&1\\1&-2\end{pmatrix}\begin{pmatrix}2&-1\\1&2\end{pmatrix}^{-1} = \begin{pmatrix}2&1\\1&-2\end{pmatrix}\frac{1}{4-(-1)}\begin{pmatrix}2&1\\-1&2\end{pmatrix} = \boxed{\frac{1}{5}\begin{pmatrix}3&4\\4&-3\end{pmatrix}}$;

(d) The formula is $A_{rot\theta} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$; With $\theta = \frac{5\pi}{6}$: $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$; $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$;
Then, $\boxed{A_4 = \frac{1}{2}\begin{pmatrix}-\sqrt{3} & -1\\ 1 & -\sqrt{3}\end{pmatrix}}$;

(e) The line is spanned by $v = \begin{pmatrix}3\\-5\end{pmatrix}$; let $w = \begin{pmatrix}5\\3\end{pmatrix}$; Observe that $v \perp w$ and $\{v, w\}$ is a basis for $\mathbb{R}^2$;
Since T_5 is an orthogonal projection: $T(v) = v$ and $T(w) = 0$;
$\therefore A_5\begin{pmatrix}3&5\\-5&3\end{pmatrix} = \begin{pmatrix}3&0\\-5&0\end{pmatrix}$; $A_5 = \begin{pmatrix}3&0\\-5&0\end{pmatrix}\begin{pmatrix}3&5\\-5&3\end{pmatrix}^{-1} = \begin{pmatrix}3&0\\-5&0\end{pmatrix}\frac{1}{9-(-25)}\begin{pmatrix}3&-5\\5&3\end{pmatrix} = \boxed{\frac{1}{34}\begin{pmatrix}9&-15\\-15&25\end{pmatrix}}$;

(f) Use the formula $A_6 = \frac{1}{\langle N,N\rangle}\left(\langle N,N\rangle I_3 - 2NN^T\right)$ for normal vector N. For T_6: $N = \begin{pmatrix}1\\2\\-3\end{pmatrix}$;

$\langle N,N\rangle = \begin{pmatrix}1\\2\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix} = 1+4+9 = 14$; $-2NN^T = -2\begin{pmatrix}1\\2\\-3\end{pmatrix}(1\ 2\ -3) = -2\begin{pmatrix}1&2&-3\\2&4&-6\\-3&-6&9\end{pmatrix} = \begin{pmatrix}-2&-4&6\\-4&-8&12\\6&12&-18\end{pmatrix}$;

$\langle N,N\rangle I_3 - 2NN^T = \begin{pmatrix}-2+14 & -4 & 6\\ -4 & -8+14 & 12\\ 6 & 12 & -18+14\end{pmatrix} = \begin{pmatrix}12&-4&6\\-4&6&12\\6&12&-4\end{pmatrix}$; $A_6 = \frac{1}{14}\begin{pmatrix}12&-4&6\\-4&6&12\\6&12&-4\end{pmatrix} = \boxed{\frac{1}{7}\begin{pmatrix}6&-2&3\\-2&3&6\\3&6&-2\end{pmatrix}}$;

Alternate Method: Use change of basis method with $T(N) = -N$, $T(d_1) = d_1$, $T(d_2) = d_2$ for direction vectors d_1, d_2 for P.
Choices for direction vectors include: $\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}$; e.g. $A_6 = \begin{pmatrix}-1&-2&3\\-2&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&-2&3\\2&1&0\\-3&0&1\end{pmatrix}^{-1}$;

(g) Use the formula $A_7 = \frac{1}{\langle N,N\rangle}\left(\langle N,N\rangle I_3 - NN^T\right)$ with normal vector N. For T_7: $N = \begin{pmatrix}1\\-2\\3\end{pmatrix}$;

$\langle N,N\rangle = \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = 1+4+9 = 14$; $(-1)NN^T = (-1)\begin{pmatrix}1\\-2\\3\end{pmatrix}(1\ -2\ 3) = (-1)\begin{pmatrix}1&-2&3\\-2&4&-6\\3&-6&9\end{pmatrix} = \begin{pmatrix}-1&2&-3\\2&-4&6\\-3&6&-9\end{pmatrix}$;

$\langle N,N\rangle I_3 - NN^T = \begin{pmatrix}-1+14 & 2 & -3\\ 2 & -4+14 & 6\\ -3 & 6 & -9+14\end{pmatrix} = \begin{pmatrix}13&2&-3\\2&10&6\\-3&6&5\end{pmatrix}$; $\boxed{A_7 = \frac{1}{14}\begin{pmatrix}13&2&-3\\2&10&6\\-3&6&5\end{pmatrix}}$

Alternate Method: Use the change-of-basis method with $T(N) = 0$, $T(d_1) = d_1$, $T(d_2) = d_2$ for direction vectors d_1, d_2 for P.
Choices for d_1, d_2 include: $\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\3\\2\end{pmatrix}$; e.g. $A_7 = \begin{pmatrix}0&2&3\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&2&3\\-2&1&0\\3&0&-1\end{pmatrix}^{-1}$;

7. Determine if the following sets of vectors are linearly independent. If the set is linearly dependent, determine a linearly independent spanning set.

(a) $V_1 = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

(b) $V_2 = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}$

(c) $V_3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$

(d) $V_4 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} \right\}$

(e) $V_5 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \right\}$

(f) $V_6 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix} \right\}$

Denote the lin. ind. spanning sets as W_i.

(a) $\det\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = (1)(3) - 2(4) = 3 - 8 = -5 \neq 0$; $\boxed{\therefore V_1 \text{ is linearly independent.}}$

(b) $\begin{pmatrix} 1 & -2 & -1 \\ -2 & 6 & 4 \end{pmatrix} \xrightarrow{2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \end{pmatrix}$; V_2 is linearly dependent. Keep vectors 1 and 2. $\boxed{W_2 = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \end{pmatrix} \right\}}$

(c) $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow[-3R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{-2R_2 + R_3 \mapsto R_3} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$;

V_3 is lin. dependent. Keep first two vectors. $\boxed{W_3 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}}$;

(d) Method 1: $\begin{pmatrix} 1 & -4 \\ 2 & 5 \\ -3 & 6 \end{pmatrix} \xrightarrow[3R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & -4 \\ 0 & 13 \\ 0 & -6 \end{pmatrix}$; Each column has a pivot. $\boxed{V_4 \text{ is linearly independent.}}$

Method 2: $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}$; $\boxed{\therefore V_4 \text{ is lin. independent.}}$

(e) V_5 has 4 vectors but is a set in $\mathbb{R}^3$. Since $4 \geq 3+1$, V_5 is linearly dependent.

$\begin{pmatrix} 1 & -4 & 7 & -1 \\ 2 & 5 & -8 & -2 \\ -3 & 6 & 9 & 3 \end{pmatrix} \xrightarrow[3R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & -4 & -1 & -1 \\ 0 & 13 & -22 & 0 \\ 0 & -6 & 30 & 0 \end{pmatrix} \xrightarrow{\frac{6}{13}R_2 + R_3 \mapsto R_3} \begin{pmatrix} 1 & -4 & 7 & -1 \\ 0 & 13 & -22 & 0 \\ 0 & 0 & \underbrace{\frac{258}{13}}_{\text{nonzero}} & 0 \end{pmatrix}$; Keep the first 3 vectors.

$\boxed{W_5 = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ -8 \\ 9 \end{pmatrix} \right\}}$;

(f) Method 1. $\begin{pmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ -3 & 6 & 9 \end{pmatrix} \xrightarrow[3R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 1 & -4 & 7 \\ 0 & 13 & -22 \\ 0 & -6 & 30 \end{pmatrix} \xrightarrow{\frac{6}{13}R_2 + R_3 \mapsto R_3} \begin{pmatrix} 1 & -4 & 7 \\ 0 & 13 & -22 \\ 0 & 0 & \frac{258}{13} \end{pmatrix}$; $\boxed{V_6 \text{ is lin. independent.}}$

Method 2. From part (e), $V_6 = W_5$ has been shown to be linearly independent.

8. Find the determinant of the following matrices.

(a) $\mathbf{B}_1 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ -2 & -8 & -10 \end{pmatrix}$

(b) $\mathbf{B}_2 = \begin{pmatrix} 3 & 4 & -5 \\ 4 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix}$

(c) $\mathbf{B}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\mathbf{B}_4 = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

(a) Method 1: Direct Calculation by Laplace Expansion.

$$\det\begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ -2 & -8 & -10 \end{pmatrix} = 2\det\begin{pmatrix} 2 & 3 \\ -8 & -10 \end{pmatrix} + (-1)(2)\det\begin{pmatrix} 1 & 3 \\ -2 & -10 \end{pmatrix} + (2)\det\begin{pmatrix} 1 & 2 \\ -2 & -8 \end{pmatrix}$$

$$= (2)[-20-(-24)] + (-2)[-10-(-6)] + (2)[(-8)-(-4)] = 2(4) + (-2)(-4) + 2(-4) = 8+8-8 = \boxed{8};$$

Method 2: Simplify using Gaussian Elimination.

$$B_1 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 3 \\ -2 & -8 & -10 \end{pmatrix} \underset{\text{swap, } m=(-1)}{\xrightarrow{R_1 \leftrightarrow R_2}} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ -2 & -8 & -10 \end{pmatrix} \xrightarrow[2R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \underset{m=-1 \text{ (no change)}}{\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -4 \\ 0 & -4 & -4 \end{pmatrix}} \xrightarrow{-2R_2 + R_3 \mapsto R_3} \underset{m=-1 \text{ (no change)}}{\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{pmatrix}};$$

$$\det(B_1) = m\det\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{pmatrix} = (-1)(1)(-2)(4) = \boxed{8};$$

(b) Method 1. Direct Calculation.

$$\det(B_2) = \det\begin{pmatrix} 3 & 4 & -5 \\ 4 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix} = (3)\det\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix} + (-1)(4)\det\begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix} + (1)\det\begin{pmatrix} 4 & -5 \\ 3 & -2 \end{pmatrix}$$

$$= (3)[-3-(-2)] + (-4)[-4-(-5)] + (1)[-6-(-15)] = (3)(-1) + (-4)(1) + (1)(7) = -3-4+7 = \boxed{0};$$

Method 2: Simplify first by Gaussian Elimination.

$$B_2 = \begin{pmatrix} 3 & 4 & -5 \\ 4 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow[\text{swap, } m=(-1)]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & -1 \\ 4 & 3 & -2 \\ 3 & 4 & -5 \end{pmatrix} \xrightarrow[-3R_1+R_3 \mapsto R_3]{-4R_1+R_2 \mapsto R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_2+R_3 \mapsto R_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix};$$

$m=-1$, no change $\qquad$ $m=-1$, no change

Option 2.1: The columns of B_2 are lin. dependent $\Leftrightarrow$ $\det(B_2) = \boxed{0};$

Option 2.2: $\det(B_2) = m\det\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)(1)(-1)(0) = \boxed{0};$

(c) Method 1: Direct Calculation. $\det(B_3) = \det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0)(\cdots) + (-1)(1)\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (0)(\cdots) = \boxed{-1};$

Method 2: $B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[m=-1]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_3;\ \det(B_3) = m\det(I_3) = (-1)(1) = \boxed{-1};$

(d) $\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = (4)(8)-(5)(7) = 32-35 = \boxed{-3};$

9. For each of the following matrices **M**, determine the following:

1. the **characteristic polynomial** of **M**,
2. the **real eigenvalues** of **M**,
3. the **eigenvectors** corresponding to the real eigenvalues of **M**

(a) $\mathbf{M}_1 = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$

(b) $\mathbf{M}_2 = -\frac{1}{5}\begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix}$

(c) $\mathbf{M}_3 = \begin{pmatrix} -20 & 11 \\ 4 & 0 \end{pmatrix}$

(d) $\mathbf{M}_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(e) $\mathbf{M}_5 = \begin{pmatrix} -2 & -1 & 8 \\ 12 & 6 & -16 \\ 6 & -1 & 0 \end{pmatrix}$

(f) $\mathbf{M}_6 = \begin{pmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{pmatrix}$

(g) $\mathbf{M}_7 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$

(a) $M_1 = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix};$

$$p(\lambda) = \det(M_1 - \lambda I_2) = \det\begin{pmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - (4)(-4) = 9-6\lambda+\lambda^2+16 = \boxed{\lambda^2-6\lambda+25};$$

Discriminant: $b^2-4ac = (-6)^2 - 4(1)(25) = 36-100 < 0$; $\boxed{\text{No real eigenvalues.}}$

(b) $M_2 = -\frac{1}{5}\begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix};$

$$p(\lambda) = \det(M_2 - \lambda I_2) = \left(-\tfrac{1}{5}\right)^2 \det(-5M+5\lambda I_2) = \tfrac{1}{25}\det\begin{pmatrix} 1+5\lambda & 3 \\ 18 & 4+5\lambda \end{pmatrix}$$

$$= \tfrac{1}{25}[(1+5\lambda)(4+5\lambda)-(18)(3)] = \tfrac{1}{25}[4+5\lambda+20\lambda+25\lambda^2-54] = \tfrac{1}{25}(25\lambda^2+25\lambda-50) = \boxed{\lambda^2+\lambda-2};$$

$p(\lambda) = \lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) = 0;$ $\boxed{\lambda = 1, -2};$

Next, solve the system: $(M_2-\lambda I_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; Equivalently, solve $-\cancel{\tfrac{1}{5}}(-5M_2+5\lambda I_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\cancel{\tfrac{1}{5}}\begin{pmatrix} 0 \\ 0 \end{pmatrix};$

For $\lambda=1$: $-5M_2+5(1)I_2 = \begin{pmatrix}1+5 & 3\\ 18 & 4+5\end{pmatrix} = \begin{pmatrix}6 & 3\\ 18 & 9\end{pmatrix} \xrightarrow{-3R_1+R_2 \mapsto R_2} \begin{pmatrix}6 & 3\\ 0 & 0\end{pmatrix}$; x_2 is a free variable.

By back-sub: $6x_1+3x_2=0$; $x_1=-\frac{1}{2}x_2$; Let $x_2=2$; Eigenvector: $\boxed{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-1\\ 2\end{pmatrix}}$;

For $\lambda=-2$: $-5M_2+5(-2)I_2 = \begin{pmatrix}1-10 & 3\\ 18 & 4-10\end{pmatrix} = \begin{pmatrix}-9 & 3\\ 18 & -6\end{pmatrix} \xrightarrow{2R_1+R_2 \mapsto R_2} \begin{pmatrix}-9 & 3\\ 0 & 0\end{pmatrix}$; x_2 is a free variable.

By back-sub: $-9x_1+3x_2=0$; $x_1=\frac{1}{3}x_2$; Let $x_2=3$; Eigenvector $\boxed{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}1\\ 3\end{pmatrix}}$,

(c) $M_3=\begin{pmatrix}-20 & 11\\ 4 & 0\end{pmatrix}$;

$p(\lambda)=\det\begin{pmatrix}-20-\lambda & 11\\ 4 & -\lambda\end{pmatrix} = -\lambda(-20-\lambda)-(4)(11) = \boxed{\lambda^2+20\lambda-44}$;

$\lambda^2+20\lambda-44=(\lambda+22)(\lambda-2)=0$; $\boxed{\lambda=2,-22}$; Next: Solve $(M_3-\lambda I_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ for each λ.

For $\lambda=2$: $M_3-\lambda I_3=\begin{pmatrix}-20-2 & 11\\ 4 & -2\end{pmatrix}=\begin{pmatrix}-22 & 11\\ 4 & -2\end{pmatrix} \xrightarrow{\frac{1}{11}R_1 \mapsto R_1} \begin{pmatrix}-2 & 1\\ 4 & -2\end{pmatrix} \xrightarrow{2R_1+R_2 \mapsto R_2} \begin{pmatrix}-2 & 1\\ 0 & 0\end{pmatrix}$; x_2 is free.

By back-sub: $-2x_1+x_2=0$; $x_1=\frac{1}{2}x_2$; Let $x_2=2$: Eigenvector $\boxed{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}}$;

For $\lambda=-22$: $M_3-\lambda I_3=\begin{pmatrix}-20+22 & 11\\ 4 & 22\end{pmatrix}=\begin{pmatrix}2 & 11\\ 4 & 22\end{pmatrix} \xrightarrow{-2R_1+R_2 \mapsto R_2} \begin{pmatrix}2 & 11\\ 0 & 0\end{pmatrix}$; x^2 is free.

By back-sub: $2x_1+11x_2=0$; $x_1=-\frac{11}{2}x_2$; Let $x_2=2$; Eigenvector $\boxed{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-11\\ 2\end{pmatrix}}$;

(d) $M_4=\begin{pmatrix}2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 4\end{pmatrix}$;

Char. Poly: $p(\lambda)=\det(A-\lambda I_3)=\det\begin{pmatrix}2-\lambda & 0 & 0\\ 0 & -3-\lambda & 0\\ 0 & 0 & 4-\lambda\end{pmatrix}=\boxed{(2-\lambda)(-3-\lambda)(4-\lambda)}$;

Eigenvalues: $\boxed{\lambda=2,-3,4.}$

Eigenvectors: For $\lambda=2$: $M_4-\lambda I_3=\begin{pmatrix}2-2 & 0 & 0\\ 0 & -3-2 & 0\\ 0 & 0 & 4-2\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & -5 & 0\\ 0 & 0 & 2\end{pmatrix}$;

Back Sub: x_1 is free.
$-5x_2=0$; $x_2=0$; Let $x_1=1$. Eigenvector: $\boxed{\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}}$;
$2x_3=0$; $x_3=0$;

Similar calculations for $\lambda=-3$: $\boxed{\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}}$ and $\lambda=4$: $\boxed{\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}}$;

(e) $M_5=\begin{pmatrix}-2 & -1 & 8\\ 12 & 6 & -16\\ 6 & -1 & 0\end{pmatrix}$;

Char Poly. $p(\lambda)=\det(M_5-\lambda I_3)=\det\begin{pmatrix}-2-\lambda & -1 & 8\\ 12 & 6-\lambda & -16\\ 6 & -1 & -\lambda\end{pmatrix}$

$=(-2-\lambda)\det\begin{pmatrix}6-\lambda & -16\\ -1 & -\lambda\end{pmatrix}+(-1)(-1)\det\begin{pmatrix}12 & -16\\ 6 & -\lambda\end{pmatrix}+(8)\det\begin{pmatrix}12 & 6-\lambda\\ 6 & -1\end{pmatrix}$

$=(-2-\lambda)[-6\lambda+\lambda^2-16]+[-12\lambda+96]+(8)[-12-36+6\lambda]$

$= 12\lambda - 2\lambda^2 + 32 + 6\lambda^2 - \lambda^3 + 16\lambda - 12\lambda + 96 - 384 + 48\lambda$

$= -\lambda^3 + 4\lambda^2 + 64\lambda - 256 = -\lambda^3 + 4\lambda^2 + 4^3\lambda - 4^4$

$= (-\lambda^2)(\lambda-4) + 4^3(\lambda-4) \; \boxed{= (\lambda-4)(-\lambda^2+64)};$

Eigenvalues: $\boxed{\lambda = 4, \pm 8};$

Eigenvectors: For $\lambda = 4$: $M_5 - 4I_3 = \begin{pmatrix} -2-4 & -1 & 8 \\ 12 & 6-4 & -16 \\ 6 & -1 & -4 \end{pmatrix} = \begin{pmatrix} -6 & -1 & 8 \\ 12 & 2 & -16 \\ 6 & -1 & -4 \end{pmatrix} \xrightarrow[R_1 + R_3 \mapsto R_3]{2R_1 + R_2 \mapsto R_2}$

$\begin{array}{c} \begin{pmatrix} -6 & -1 & 8 \\ 0 & 0 & 0 \\ 0 & -2 & 4 \end{pmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \end{array}$; x_3 is a free variable; Back Sub: $-6x_1 - x_2 + 8x_3 = 0$; $-2x_2 + 4x_3 = 0$;

$x_2 = 2x_3$; $-6x_1 = x_2 - 8x_3 = 2x_3 - 8x_3 = -6x_3$; $x_1 = x_3$;

Let $x_3 = 1$. Eigenvector: $\boxed{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$; for $\lambda = 4$;

For $\lambda = 8$: $M_5 - 8I_3 = \begin{pmatrix} -2-8 & -1 & 8 \\ 12 & 6-8 & -16 \\ 6 & -1 & -8 \end{pmatrix} = \begin{pmatrix} -10 & -1 & 8 \\ 12 & -2 & -16 \\ 6 & -1 & -8 \end{pmatrix} \begin{matrix} 6R_1 \mapsto R_1 \\ 5R_2 \mapsto R_2 \\ 10R_3 \mapsto R_3 \end{matrix} \rightarrow \begin{pmatrix} -60 & -6 & 48 \\ 60 & -10 & -80 \\ 60 & -10 & -80 \end{pmatrix}$

$\xrightarrow{-R_2 + R_3} \begin{pmatrix} -60 & -6 & 48 \\ 60 & -10 & -80 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + R_2 \mapsto R_2} \begin{pmatrix} -60 & -6 & 48 \\ 0 & -16 & -32 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} -\frac{1}{6}R_1 \mapsto R_1 \\ -\frac{1}{16}R_2 \mapsto R_2 \end{matrix} \rightarrow$

$\begin{pmatrix} 10 & 1 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; x_3 is free. Back Sub: $10x_1 + x_2 - 8x_3 = 0$; $x_2 + 2x_3 = 0$;

$x_2 = -2x_3$; $10x_1 = -x_2 + 8x_3 = -(-2x_3) + 8x_3 = 10x_3$; $x_1 = x_3$;

Choose $x_3 = 1$; Eigenvalue: $\boxed{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$; for $\lambda = 8$

For $\lambda = -8$: $M_5 + 8I_3 = \begin{pmatrix} -2+8 & -1 & 8 \\ 12 & 6+8 & -16 \\ 6 & -1 & 8 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 8 \\ 12 & 14 & -16 \\ 6 & -1 & 8 \end{pmatrix} \xrightarrow[-R_1 + R_3 \mapsto R_3]{-2R_1 + R_2 \mapsto R_2} \begin{pmatrix} 6 & -1 & 8 \\ 0 & 16 & -32 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{16}R_2 \mapsto R_2}$

$\begin{pmatrix} 6 & -1 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$; x_3 is free. Back Sub: $6x_1 - x_2 + 8x_3 = 0$; $x_2 - 2x_3 = 0$;

$x_2 = 2x_3$; $6x_1 = x_2 - 8x_3 = 2x_3 - 8x_3 = -6x_3$; $x_1 = -x_3$;

Choose $x_3 = 1$; Eigenvalue: $\boxed{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$ for $\lambda = -8$

(f) See lecture 4B.

(g) $M_7 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$;

Char Poly: $p(\lambda) = \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 0 & 2 & 1-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} + (0)(\ldots) \boxed{= (2-\lambda)[(1-\lambda)^2 + 4]};$

Eigenvalues: $p(\lambda) = 0$: $2 - \lambda = 0$; $\lambda = 2$;

$(1-\lambda)^2 + 4 = 0$; $(1-\lambda)^2 = -4$; No real solutions. Eigenvalue: $\boxed{\lambda = 2};$

Eigenvector for $\lambda = 2$:

$M_7 - 2I_3 = \begin{pmatrix} 2-2 & 0 & 0 \\ 0 & 1-2 & -2 \\ 0 & 2 & 1-2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{2R_2 + R_3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -3 \end{pmatrix}$; x_1 is free. Back Sub: $-x_2 - 2x_3 = 0$; $-3x_3 = 0$

$x_3 = 0$; $-x_2 - 2(0) = 0$; $x_2 = 0$; Choose $x_1 = 1$: Eigenvalue: $\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ for $\lambda = 2$