Name: Answer Key

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

(1) Consider the sequence $a_n = 5^{-n}$ for $n \in \mathbb{Z}_{\geq 0}$.

(a) Find the first 5 terms of a_n. That is, find a_0, a_1, a_2, a_3, and a_4.

(b) Find the 4^{th} partial sum S_4 of (a_n).

(c) Determine if $\sum_{n=0}^{\infty} a_n$ converges or diverges. State the result you are using to conclude convergence or divergence. If it converges, evaluate the sum.

(a) $a_0 = 1$; $a_1 = \frac{1}{5}$; $a_2 = \frac{1}{25}$; $a_3 = \frac{1}{125}$; $a_4 = \frac{1}{625}$;

(b) $S_4 = \sum_{n=0}^{4} a_n$; Method 1: $S_4 = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} = \boxed{\frac{781}{625}}$;

Method 2: $S_4 = a\left(\frac{r^{n_0} - r^{n+1}}{1-r}\right) = (1)\left(\frac{(\frac{1}{5})^0 - (\frac{1}{5})^5}{1-\frac{1}{5}}\right) = \boxed{\frac{781}{625}}$;

(c) $\sum_{n=0}^{\infty} 5^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$ is a geometric series with $r = \frac{1}{5}$; Since $|r| = \frac{1}{5} < 1$, $\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$ converges

and $\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n = (1)\left(\frac{1}{5}\right)^0\left(\frac{1}{1-\frac{1}{5}}\right) = \frac{1}{\frac{4}{5}} = \boxed{\frac{5}{4}}$;

(2) Consider the sequence $a_n = \sin(\pi n + \frac{\pi}{2})$ for $n \in \mathbb{Z}_{\geq 0}$.

(a) Find the first 4 terms a_0, a_1, a_2, and a_3.

(b) Determine $\lim_{n\to\infty} a_n$.

(c) Determine if $\sum_{n=0}^{\infty} a_n$ converges. State the result you are using to conclude convergence or divergence. If it converges, evaluate the sum.

Simplifying first:

If n is even, i.e. $n = 2k$ for some $k \in \mathbb{Z}$: $a_{2k} = \sin\left(\pi(2k) + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$;

If n is odd, i.e. $n = 2k+1$ for some $k \in \mathbb{Z}$: $a_{2k+1} = \sin\left(\pi(2k+1) + \frac{\pi}{2}\right) = \sin\left(2\pi k + \pi + \frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$;

i.e. $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases} = (-1)^n$;

(a) $a_0 = 1$; $a_1 = -1$; $a_2 = 1$; $a_3 = -1$;

(b) $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(-1)^n = \text{DNE}$. One justification: $\lim_{n\to\infty}(a_{2n}) = \lim_{n\to\infty}(1) = 1 \neq \lim_{n\to\infty}(a_{2n+1}) = \lim_{n\to\infty}(-1) = -1$;

(c) Method 1: $\sum a_n = \sum (-1)^n$ is a geometric series with $r = -1$.
Since $|r| = 1 \geq 1$, $\sum a_n$ diverges.

Method 2: Since $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges by the Divergence Test.

This was removed due to an error.

(3) Consider the sequence $a_n = \cancel{(4n^2 + 6n + 2)^{-1}}$ for $n \in \mathbb{Z}_{\geq 0}$. Intended: $a_n = 2(4n^2 + 8n + 3)^{-1}$;

(a) Find the first 3 partial sums S_0, S_1, S_2 of a_n. Note that the index starts at $n_0 = 0$.

(b) Find a closed form for the n^{th} partial sum S_n. i.e. express S_n as a function of n.

(c) Determine if $\sum_{n=0}^{\infty} a_n$ converges. State the result you are using to conclude convergence or divergence. If it converges, evaluate the sum.

(a) $a_0 = \frac{2}{3}$; $a_1 = \frac{2}{15}$; $a_2 = \frac{2}{35}$; $S_0 = a_0 = \frac{2}{3}$; $S_1 = a_0 + a_1 = \frac{4}{5}$; $S_2 = a_0 + a_1 + a_2 = \frac{6}{7}$;

(b) By what's been covered in class, closed forms of partial sums are given only for geometric series and telescoping series. Guess: This may be a telescoping series. Do a partial fraction decomposition.
Method 1. Find a, b such that $ab = (4)(3) = 12$ and $a + b = 8$; $a = 6, b = 2$ work.

$$4n^2 + 8n + 3 = 4n^2 + 6n + 2n + 3 = 2n(2n+3) + (2n+3) = (2n+1)(2n+3);$$

Method 2. Complete the square with $(an+b)^2 = a^2n^2 + 2abn + b^2$;
With $4n^2 + 8n + b^2$: $a^2 = 4$; $a = 2$; $2ab = 8$; $b = 2$; $b^2 = 4$;

$$4n^2 + 8n + 3 = (4n^2 + 8n + 4) + 3 - 4 = (2n+2)^2 - 1 = \left[(2n+2) - 1\right]\left[(2n+2) + 1\right]$$
$$= (2n+1)(2n+3);$$

PFD: $\frac{2}{(2n+1)(2n+3)} = \frac{A}{2n+1} + \frac{B}{2n+3}$; $2 = A(2n+3) + B(2n+1)$;
let $n = -\frac{3}{2}$: $2 = B(2(-\frac{3}{2}) + 1) = B(-3+1) = -2B$; $B = -1$;
let $n = -\frac{1}{2}$: $2 = A(2(-\frac{1}{2}) + 3) = A(-1+3) = 2A$; $A = 1$;

Then, $a_n = \frac{1}{2n+1} + (-1)\frac{1}{2n+3}$; let $b(n) = \frac{1}{2n+1}$; Observe that $b(n+1) = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$;

Therefore, $\sum_{n=0}^{\infty} a_n$ is a telescoping sum with $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left[b(n) - b(n+1)\right]$;

and $S_n = \sum_{k=0}^{n} a_n = b(0) - b(n+1) = \frac{1}{2(0)+1} - \frac{1}{2n+3} = \boxed{1 - \frac{1}{2n+3}}$;

(c) Since $\lim_{n\to\infty} b(n) = \lim_{n\to\infty}\left(\frac{1}{2n+1}\right) = 0$: $\sum_{n=0}^{\infty} a_n$ converges with

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+3}\right) = \boxed{1};$$