HW2B. Written Homework 2B. **Due Week 2 Friday 11:59PM**

Name: Answer Key

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

Q1

(1) Use the **Integral Test** to determine the convergence of the following series. If the Integral Test is inapplicable, state at least one condition that it fails to satisfy.

Note that there may be other methods to determine the convergence of the following series. However, this worksheet tests your knowledge and understanding of the Integral Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\pi n + \frac{\pi}{2}\right)$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n-3}{n^2-3n+4}\right)$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2-3n+4}{2n-3}\right)$

Note. The Limit Comparison Test is usually used for series like these.

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\arctan n}{n^2+1}\right)$

(e) $\displaystyle\sum_{k=1}^{\infty} ke^{-3k^2}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$

(a) $f_a(x) = \sin\left(\pi x + \frac{\pi}{2}\right)$ is not positive and not decreasing on $[a,\infty)$ for any $a \in \mathbb{R}$. IT is not applicable.

(b) $f_b(x) = \dfrac{2x-3}{x^2-3x+4}$; Check the conditions.

(i) x^2-3x+4 has no real roots since $\Delta = (-3)^2 - 4(1)(4) = 9-16 = -7 < 0$. So, $f_b(x)$ is continuous on $\mathbb{R}$.

(ii) By the sign chart method: $f_b(x)$ has no discontinuities. For zeros: $f_b(x)=0$; $2x-3=0$; $x=\frac{3}{2}$; Testing $x=2 \in \left(\frac{3}{2},\infty\right)$: $f_b(2)$ is $(+)$. Then, $f_b(x)$ is positive on $[3,\infty) \subseteq \left(\frac{2}{3},\infty\right)$.

(iii) $f_b'(x) = (x^2-3x+4)^{-2}\left[(x^2-3x+4)(2) - (2x-3)(2x-3)\right] = (x^2-3x+4)^{-2}\left[-2x^2+6x+1\right]$;

Do a sign chart on $f_b'(x)$. Since $f_b'(x)$ is continuous on $\mathbb{R}$, we only need to find the zeros of $f_b'(x)$. Equivalently, find all $x \in \mathbb{R}$ such that $-2x^2+6x+1=0$;

By the quadratic formula: $x_{1,2} = \dfrac{1}{2(-2)}\left(-6 \pm \sqrt{(6)^2-4(-2)(1)}\right) = 0.177, 2.823$;

Chart:

		0.177		2.823	
$f_b(x)$	$\searrow$		$\nearrow$		$\searrow$
$f_b'(x)$	$(-)$		$(+)$		$(-)$
	Try $x=0$: $f_b'(0) = -0.0625$		Try $x=1$: $f_b'(1) = 0.75$		Try $x=3$: $f_b'(3) = -0.0625$

Therefore, $f_b(x)$ is decreasing for $x > 2.823$;

Therefore, we can apply the Integral Test on $\displaystyle\sum_{n=3}^{\infty}\left(\frac{2n-3}{n^2-3n+4}\right)$ with $f_b(x) = \dfrac{2x-3}{x^3-3x+4}$; Observe the change in starting index.

Then, $\displaystyle\int_3^{\infty} \frac{2x-3}{x^2-3x+4}\,dx = \lim_{b\to\infty}\int_3^b \frac{2x-3}{x^2-3x+4}\,dx \underset{du=2x-3\,dx}{\overset{u=x^2-3x+4}{=}} \lim_{b\to\infty}\int_{x=3}^{x=b}\frac{1}{u}\,du = \lim_{b\to\infty}\Big[\ln|u|\Big]_{x=3}^{x=b}$

$\displaystyle= \lim_{b\to\infty}\left[\ln(b^2-3b+4) - \ln(3^2-3(3)+4)\right] = \lim_{b\to\infty}\ln(b^2-3b+4) - \ln(4) = \infty$, i.e. diverges.

By the Integral Test, $\displaystyle\sum_{n=3}^{\infty}\frac{2n-3}{n^2-3n+4}$ diverges. Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{2n-3}{n^2-3n+4}$ also diverges.

(c) $f_c(x) = \dfrac{x^2-3x+4}{2x-3}$; Since $\lim\limits_{x\to\infty} f_c(x) = \lim\limits_{x\to\infty} \dfrac{x^2-3x+4}{2x-3} = \infty$, $f_c(x)$ cannot be decreasing for any $[a,\infty)$.

Equivalently: $f_c'(x) = (2x-3)^{-2}\left[(2x-3)(2x-3) - (x^2-3x+4)(2)\right] = (2x-3)^{-2}(2x^2-6x+1)$;

Discontinuities: $2x-3=0$; $x = \frac{3}{2} = 1.5$;

Zeros: $2x^2-6x+1$; $x_{1,2} = [2(2)]^{-1}\left[6 \pm \sqrt{(-6)^2-4(2)(1)}\right] \approx 0.177, 2.823$;

Sign Chart:

		0.177		1.5		2.823	
$f_c(x)$	$\nearrow$		$\searrow$		$\searrow$		$\nearrow$
$f_c'(x)$	$(+)$		$(-)$		$(-)$		$(+)$

$f_c'(0) = 0.\overline{1}$ $\quad f_c'(1) = -3$ $\quad f_c'(2) = -3$ $\quad f_c'(3) = 0.\overline{1}$

Therefore, $f_c(x)$ is increasing on $[3,\infty)$ and the Integral Test is not applicable.

Remark: $f_c(x)$ is continuous and positive on $[2,\infty)$.

(d) $f_d(x) = \dfrac{\arctan(x)}{1+x^2}$;

(i) $1+x^2$ has no zeroes and $\arctan(x)$ is continuous on $\mathbb{R}$. $\therefore f_d(x)$ is continuous on $\mathbb{R}$.

(ii) For $x \geq 0$, $\arctan(x) \in [0, \frac{\pi}{2})$. For all $x \in \mathbb{R}$, $1+x^2$ is positive. $\therefore f_d(x)$ is positive on $[0,\infty)$.

(iii) $f_d'(x) = (1+x^2)^{-2}\left[(1+x^2)(1+x^2)^{-1} - \arctan(x)(2x)\right] = (1+x^2)^{-2}\left[1-2x\arctan(x)\right]$;

Assuming $x \geq 1$, $\arctan(x) \geq \frac{\pi}{4} \approx 0.785$; Then, $2x\arctan(x) \geq 2(1)\arctan(x) \geq 2(\frac{\pi}{4}) = \frac{\pi}{2} = 1.507\ldots > 1$.

Therefore, $1-2x\arctan(x)$ is negative for $x \geq 1$. Since $(1+x^2)$ is always positive, $f_d'(x)$ is negative for $x \in [1,\infty)$.

$\therefore f_d(x)$ is decreasing on $[1,\infty)$.

We can apply the Integral Test on $\sum\limits_{n=1}^{\infty} \dfrac{\arctan(n)}{1+n^2}$;

Then, $\displaystyle\int_1^{\infty} \frac{\arctan(x)}{1+x^2}\,dx = \lim_{b\to\infty} \int_1^{\infty} \frac{\arctan(x)}{1+x^2}\,dx \underset{du=(1+x^2)^{-1}dx}{\overset{u=\arctan(x)}{=}} \lim_{b\to\infty}\int_{x=1}^{x=b} u\,du = \lim_{b\to\infty}\left[\frac{1}{2}u^2\right]_{x=1}^{x=b}$

$\displaystyle = \frac{1}{2}\lim_{b\to\infty}\left[(\arctan(b))^2 - (\arctan(1))^2\right] = \frac{1}{2}\left[\left(\lim_{b\to\infty}\arctan(b)\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$

$\displaystyle = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \frac{\pi^2}{16}\right] < \infty$. By the Integral Test, $\sum\limits_{n=1}^{\infty}\dfrac{\arctan(n)}{1+n^2}$ converges!

(e) $\sum\limits_{k=1}^{\infty} ke^{-3k^2}$; Let $f(x) = xe^{-3x^2}$; Restrict $x \in [1,\infty)$. Check the conditions:

① $f(x)$ is continuous on $\mathbb{R}$.

② For $x \in [1,\infty)$: x is positive; e^{-3x^2} is always positive. $\therefore xe^{-3x^2} = f(x)$ is positive on $[1,\infty)$.

③ $f'(x) = x(-6x)e^{-3x^2} + e^{-3x^2} = (1-6x^2)e^{-3x^2}$;

Method 1: For $x \geq 1$: $6x^2 \geq 6 > 1$. Then, $0 > 1-6x^2$. $\therefore f'(x)$ is negative on $[1,\infty)$.

Method 2: Use the Sign Chart method: $f'(x)$ has no discontinuities.

For zeros: $f'(x)=0$; $1-6x^2=0$; $6x^2=1$; $x^2=\frac{1}{6}$; $x = \pm\frac{1}{\sqrt{6}}$;

For the interval $(\frac{1}{\sqrt{6}},\infty)$: test $x=1 > \frac{1}{\sqrt{6}}$: $f'(x)$ is $(-)$. $\therefore f'(x)$ is negative on $[1,\infty)$.

$\therefore f(x)$ is decreasing on $[1,\infty)$.

We can apply the Integral Test.

Then, $\displaystyle\int_1^{\infty} xe^{-3x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-3x^2}\,dx = \lim_{b\to\infty}\int_{-3}^{-3b^2} -\frac{1}{6}e^u\,du = \lim_{b\to\infty}\left[-\frac{1}{6}e^u\right]_{-3}^{-3b^2}$

$\left[\begin{array}{l} u=-3x^2;\ du=-6x\,dx;\ -\frac{1}{6}du = x\,dx; \\ x=b: u=-3b^2;\ \ x=1: u=-3; \end{array}\right]$

$$= -\frac{1}{6}\lim_{b\to\infty}\left[e^{-3b^2} - e^{-3}\right] = -\frac{1}{6}\left(0 - e^{-3}\right) < \infty.$$

By the Integral Test, $\sum_{k=1}^{\infty} ke^{-3k^2}$ is convergent.

(f) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$; let $f(x) = \frac{\ln(x)}{x^2}$;

Check the conditions.

① Since for $x \in [2,\infty)$: $\ln(x) > 0$ and $x^2 > 0$, $f(x)$ is positive.

② $\ln(x)$ is continuous over $(0,\infty)$. Since $x^2 = 0$ if $x = 0$, $f(x)$ is discontinuous on $\mathbb{R}$ except at $x=0$.
$\therefore f(x)$ is continuous over $[2,\infty)$.

③ $f'(x) = \frac{x^2\left(\frac{1}{x}\right) - \ln(x)(2x)}{x^4} = \frac{x - 2x\ln(x)}{x^4} = \frac{(1-2\ln(x))x}{x^4} = \frac{1-2\ln(x)}{x^3}$;

$f'(x)$ is not defined for $(-\infty, 0]$. The zeroes of $f'(x)$: $1 - 2\ln(x) = 0$; $\ln(x) = \frac{1}{2}$; $x = \sqrt{e} \approx 1.64$;
Partition $(0,\infty)$ using $x = \sqrt{e}$: For the interval $(2,\infty)$: since $f'(e) = (-)$, $f'(x)$ is negative on $[2,\infty)$.
$\therefore f(x)$ is decreasing on $[2,\infty)$.

Applying the Integral Test:

$$\int \frac{\ln(x)}{x^2}dx \overset{\text{Integration by Parts}}{=} uv - \int v\,du = -\frac{\ln(x)}{x} - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx = -\frac{\ln(x)}{x} + \int x^{-2}dx$$

$\left[\begin{array}{l} u = \ln(x);\quad dv = x^{-2}dx; \\ du = \frac{1}{x}dx;\quad v = \frac{1}{-2+1}x^{-2+1} = -x^{-1}; \end{array}\right.$

$$= -\frac{\ln(x)}{x} + \frac{1}{-1}x^{-1} + C;$$

Then, $\int_2^{\infty} \frac{\ln(x)}{x^2}dx = \lim_{x\to\infty}\left(\frac{-\ln(x)+1}{1-x} + C\right) \overset{L'R}{=} \lim_{x\to\infty}\left(\frac{-\frac{1}{x}}{-1}\right) + C = 0 + C = C$ for some $C \in \mathbb{R}$.

Since $\int_2^{\infty}\frac{\ln(x)}{x^2}dx$ converges, $\sum_{n=2}^{\infty}\frac{\ln(n)}{n^2}$ converges by the Integral Test.

Q2 **(2)** Find all p such that the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^p}$ converges. Hint: The Integral Test can be used to identify p.

Label the series as $S(p) = \sum_{n=1}^{\infty}\frac{\ln(n)}{n^p}$; let $f(x) = \frac{\ln(x)}{x^p}$; Assume $x \in [1,\infty)$ for all cases.

First, we'll eliminate some values of p using the Divergence Test.

If $p=0$: $f(x) = \frac{\ln(x)}{x^0} = \ln(x)$; Since $\lim_{n\to\infty}\ln(x) = \infty \neq 0$: $S(0)$ diverges by the Divergence Test.

If $p \neq 0$: $\lim_{x\to\infty}\left(\frac{\ln(x)}{x^p}\right) \overset{L'R}{=} \lim_{x\to\infty}\left(\frac{1}{x}\right)\left(\frac{1}{px^{p-1}}\right) = \lim_{x\to\infty}\left(\frac{1}{px^p}\right)$;

If $p \in (-\infty, 0)$: $\lim_{x\to\infty}\left(\frac{1}{px^p}\right) = \lim_{x\to\infty}(x^k) = \infty \neq 0$ for $k = -p$. Observe that k is positive. ☆
$S(p)$ diverges for $p \in (-\infty,0)$ by the Divergence Test.

If $p \in (0,\infty)$: $\lim_{x\to\infty}\left(\frac{1}{px^p}\right) = 0$ since $\lim_{x\to\infty}(x^p) = \infty$. The Divergence Test is inconclusive. ☆

Summarizing, we now have $\begin{cases} \text{if } p \in (-\infty, 0]: & S(p) \text{ diverges} \\ \text{if } p \in (0,\infty): & \text{inconclusive} \end{cases}$;

Assume $p \in (0,\infty)$. We can check that the conditions of the Integral Test are satisfied.

(1) $\ln(x)$ is continuous on $(0,\infty)$. x^p only has one zero: $x=0$.

(2) $\ln(x)$ is positive on $(1,\infty)$ and $\ln(1)=0$. x^p is positive on $(0,\infty)$.

Since $\frac{\ln(1)}{1^p}=0$, $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^p}=\sum_{n=2}^{\infty}\frac{\ln(n)}{n^p}$. Then, we restrict our attention to $x\in[2,\infty)$.

$\therefore f(x)$ is positive on $[2,\infty)$.

(3) $f'(x)=(x^p)^{-2}\left(x^p\left(\frac{1}{x}\right)-\ln(x)(p)x^{p-1}\right)=x^{-2p}\left(x^{p-1}-p\ln(x)x^{p-1}\right)=x^{-2p}x^{p-1}\left(1-p\ln(x)\right)$;

Over $x\in[2,\infty)$: x^{-p-1} is positive. We need to find x s.t. $1-p\ln(x)<0$.

By continuity of $1-p\ln(x)$ over $[2,\infty)$, we can do the sign chart method.

$1-p\ln(x)=0$; $\ln(x)=\frac{1}{p}$; $x=e^{\frac{1}{p}}=\sqrt[p]{e}$; We can partition $[2,\infty)$ into $[2,\sqrt[p]{e})$ and $(\sqrt[p]{e},\infty)$.

For $x\in(\sqrt[p]{e},\infty)$, $f'(x)$ is negative since testing for $x=2\sqrt[p]{e}$:

$$1-p\ln(2\sqrt[p]{e})=1-p\ln(e^{2\frac{1}{p}})=1-p(2\tfrac{1}{p})\ln(e)=1-2=-1;$$

For fixed $p\in(0,\infty)$, there exists $N_p\in\mathbb{Z}$ such that $N_p\geq 2$ and $N_p\geq\sqrt[p]{e}$.

We can apply the integral test on $[N_p,\infty)$. For brevity, we'll the limit of the indefinite integral instead. There are two cases.

Case 1: Assume $p=1$.

Integral: $\int f(x)\,dx=\int\frac{\ln(x)}{x}\,dx \underset{du=\frac{1}{x}dx;}{\overset{u=\ln(x);}{=}} \int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}(\ln(x))^2+C$;

Then, $\lim_{x\to\infty}\left[\frac{1}{2}(\ln(x))^2+C\right]=\infty$. By the Integral Test, $\sum_{n=N_1}^{\infty}\frac{\ln(n)}{n}$ diverges.

Case 2: Assume $p\neq 1$. That is, $p\in(0,1)$ or $p\in(1,\infty)$.

Integral: $\int f(x)\,dx=\int\frac{\ln(x)}{x^p}\,dx=uv-\int v\,du=\frac{1}{-p+1}x^{-p+1}\ln(x)-\int\frac{1}{-p+1}x^{-p+1}\frac{1}{x}\,dx$

$$\left[\begin{array}{ll} u=\ln(x) & dv=x^{-p}dx \\ du=\frac{1}{x}dx & v=\frac{1}{-p+1}x^{-p+1}\end{array}\right]$$

$$=\frac{\ln(x)}{(-p+1)x^{p-1}}-\int\frac{1}{-p+1}x^{-p}\,dx=\frac{\ln(x)}{(-p+1)x^{p-1}}-\frac{1}{(-p+1)^2}x^{-p+1}+C=\frac{(-p+1)\ln(x)+1}{(-p+1)^2x^{p-1}}+C$$

Then, $L=\lim_{x\to\infty}\int f(x)\,dx=\lim_{x\to\infty}\frac{(-p+1)\ln(x)-1}{(-p+1)^2x^{p-1}}\overset{\text{L'H}}{=}\lim_{x\to\infty}\frac{(-p+1)\frac{1}{x}}{(-p+1)^3(-1)x^{p-2}}=K\lim_{x\to\infty}x^{-(p+1)}$

with $K=(-1)(-p+1)^{-2}$; We consider 2 cases again.

If $p\in(0,1)$: $p-1$ is negative and $\lim_{x\to\infty}x^{-(p-1)}=\infty$. $\therefore L=\infty$.

By the Integral Test, $S(p)$ diverges for $p\in(0,1)$.

If $p\in(1,\infty)$: $p-1$ is positive and $\lim_{x\to\infty}x^{p-1}=\infty$ and $\lim_{x\to\infty}x^{-(p-1)}=0$.

By the Integral Test, $S(p)$ converges for $p\in(1,\infty)$.

! Answer: $\sum_{n=1}^{\infty}\frac{\ln(n)}{n^p}$ converges for $p\in(1,\infty)$ and diverges for $p\in(-\infty,1]$;

Q2. Alternate Method. This problem can be made easier with the Comparison Test.

Note that the Limit Comparison Test will not work since the limit either goes to zero or infinity.

We'll establish two bounds: (1) For $n \geq 3$: $n > e$. Then, $\ln(n) > \ln(e) = 1$ since $\ln(x)$ is increasing on $(0, \infty)$.

(2) Recall that $\int_1^x \frac{1}{t}\,dt = \ln(x) + \ln(1) = \ln(x)$. Assume that $x \geq 1$. Then, for all $t \in [1, x]$: $\frac{1}{t} \leq 1$.

By properties of the integral: $\ln(x) = \int_1^x \frac{1}{t}\,dt \leq \int_1^x 1\,dt = x - 1 < x$; Therefore, $x > \ln(x)$ for $x \geq 1$.

For $n \geq 3$: $n > \ln(n) > 1$. Then, $\frac{1}{n^{p-1}} = \frac{n}{n^p} > \frac{\ln(n)}{n^p} > \frac{1}{n^p}$;

If $p \in (-\infty, 1]$: $\sum_{n=3}^{\infty} \frac{1}{n^p}$ diverges as a p-series since $p \geq 1$; Since for $n \geq 3$: $\frac{\ln(n)}{n^p} > \frac{1}{n^p}$, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n^p}$ diverges by CT.

If $p \in (2, \infty)$: $\sum_{n=3}^{\infty} \frac{1}{n^{p-1}}$ converges as a p-series since $p - 1 > 1$. Since for $n \geq 3$: $\frac{1}{n^{p-1}} = \frac{n}{n^p} > \frac{\ln(n)}{n^p}$, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n^p}$ converges by CT.

If $p \in (1, 2)$: The Comparison Test tells us nothing. Proceed to the Integral Test as above.

This is a typo. The Integral Test cannot be applied since $f(x) = x^3$ is increasing on $[1, \infty)$. We do not have a result about area corrected approx. if they do not pass the Integral Test.

Q3. **(3)** Let $\boxed{S = \sum_{n=1}^{\infty} 1n^3}$. Find an area corrected approximation U_N of S accurate to within 0.001.

(Note: This problem will not be graded for accuracy) Correction. Let $S = \sum_{n=1}^{\infty} \frac{1}{n^3}$;

Step 1. Confirm that the Integral Test determines that S converges.

Let $f(x) = \frac{1}{x^3}$; For $x \in [1, \infty)$: $f(x)$ is positive and continuous. $f'(x) = (-3)x^{-4}$ is negative and $f(x)$ is decreasing.

$\therefore$ The Integral Test applies on $[1, \infty)$

Then, $L = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x^3} = 0$. By the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Step 2. Find N such that $E_N = S - U_N$ is bounded above by 0.001.

Using an approximation theorem in class, $0 < E_N = S - U_N < f(N+1)$.

Since $f(x)$ is decreasing on $[1, \infty)$, it's sufficient to find x such that $f(x+1) \leq 0.001$;

Then, $f(x+1) = \frac{1}{(x+1)^3} = 0.001$; $(x+1)^3 = \frac{1}{0.001} = 1000$; $x + 1 = \sqrt[3]{1000} = 10$; $x = 10 - 1 = 9$;

Choose $N = \lceil 9 \rceil = 9$. By the Approximation Theorem, $0 < E_9 = S - U_9 < f(10) \leq 0.001$ and $U_9 = \sum_{n=1}^{9} \frac{1}{n^3}$ is accurate to within 0.001;

Step 3. Evaluate U_9.

Using a calculator: $U_9 = \sum_{n=1}^{9} \frac{1}{n^3} \approx \boxed{1.19653}$;