HW3A. Written Homework 3A. Comparison Tests. **Due Week 3 Wednesday 11:59PM**

Name: Answer Key

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

(1) For each of the series below, determine if there's a p-series or a geometric series that can be used for a Comparison Test. Explicitly show a justification as to why you can or can't use it for the Comparison Test. If it exists, apply the **Comparison Test** on the series and interpret the result.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^3+n^2}$ converges by CT

(b) $\displaystyle\sum_{n=11}^{\infty} \frac{1}{n-10}$ diverges by CT

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n+1}$ converges by CT

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$ CT is inapplicable but diverges by LCT.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$ CT is inapplicable but converges by LCT

(a) Let $a_n = \dfrac{1}{n^4+n^3+n^2}$ and let $b_n = \dfrac{1}{n^4}$;

$\sum_{n=1}^{\infty} b_n$ converges as a p-series with $p = 4 > 1$.

For $n \geq 1$: $n^4+n^3+n^2 \geq n^4$;

$$a_n = \frac{1}{n^4+n^3+n^2} \leq \frac{1}{n^4} = b_n;$$

By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let $a_n = \dfrac{1}{n-10}$ and $b_n = \dfrac{1}{n}$;

$\sum_{n=11}^{\infty} b_n$ diverges as a p-series with $p = 1 \leq 1$.

We want to show that $a_n \geq b_n$ for $n \geq 11$.

For $n \geq 11$: $0 < n-10 < n$;

$$a_n = \frac{1}{n-10} \geq \frac{1}{n} = b_n;$$

By the Comparison Test, $\sum_{n=1}^{\infty} a_n$ diverges.

(c) Let $a_n = \dfrac{1}{2^n+1}$ and $b_n = \dfrac{1}{2^n} = \left(\frac{1}{2}\right)^n$;

$\sum_{n=0}^{\infty} b_n$ converges as a geometric series with $|r| = \frac{1}{2} < 1$.

We want to show that $a_n \leq b_n$ for all $n \geq 1$.

For $n \geq 0$: $0 < 2^n < 2^n+1$;

$$b_n = \frac{1}{2^n} \geq \frac{1}{2^n+1} = a_n;$$

By the Comparison Test, $\sum_{n=0}^{\infty} a_n$ converges.

(d) Let $a_n = \dfrac{1}{2^n-1}$ and $b_n = \dfrac{1}{2^n} = \left(\frac{1}{2}\right)^n$;

$\sum_{n=0}^{\infty} b_n$ converges as a geometric series with $|r| = \frac{1}{2} < 1$.

We want to show that $a_n \leq b_n$ for all $n \geq 1$.

For $n \geq 1$: $0 < 2^n-1 < 2^n$;

$$a_n = \frac{1}{2^n-1} \geq \frac{1}{2^n} = b_n;$$

This will not work with the Comparison Test!

(e) Let $a_n = \dfrac{n+1}{n^3+n}$ and $b_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$;

$\sum_{n=1}^{\infty} b_n$ converges as a p-series with $p = 2 > 1$.

We want to show that $a_n \leq b_n$ for all $n \geq 1$.

Method 1. Equivalently, WTS that $a_n - b_n \leq 0$;

$$\text{let } f(x) = \frac{x+1}{x^3+x} - \frac{1}{x^2} = \frac{(x+1)x-(x^2+1)}{x^2(x^2+1)}$$

$$= \frac{x^2+x-x^2-1}{x^2(x^2+1)} = \frac{x-1}{x^2(x^2+1)};$$

Discontinuity @ $x=0$; Zeros @ $x=1$;

For the interval $(1,\infty)$, the sign of $f(x)$ will be constant.

Since $f(2) = (+)$, $f(x)$ is $(+)$ on $(1,\infty)$.

$\therefore$ For all $n \geq 2$: $a_n \geq b_n$.

This will not work with the Comparison Test.

Method 2. For $n \geq 2$: $0 < n^2 < n^3 < n^3+n$; $\dfrac{1}{n^3+n} \geq \dfrac{1}{n^2}$;

Since $n+1 > 1$: $\dfrac{n+1}{n^3+n} \geq \dfrac{1}{n^2}$;

This will not work with the Comparison Test.

(2) Use the **Limit Comparison Test** to determine the convergence of the following series. Identify the series $\sum b_n$ being used for the Limit Comparison Test.

Note that there may be other methods to determine the convergence of the following series. However, this problem tests your knowledge and understanding of the Comparison Test, not the Limit Comparison Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+10}$ diverges by LCT; **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges by LCT; **(e)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^4 - n^3 - n^2}$ converges by LCT; **(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ diverges by LCT;

(a) let $a_n = \frac{1}{n+10}$ and $b_n = \frac{1}{n}$;

$\sum_{n=1}^{\infty} b_n$ diverges as a p-series with $p = 1 \leq 1$.

$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{n}{n+10} = 1$.

Since $0 < L < \infty$, $\sum_{n=1}^{\infty} a_n$ also diverges by the Limit Comparison Test.

(b) let $a_n = \frac{1}{n^4 - n^3 - n^2}$ and $b_n = \frac{1}{n^4}$;

$\sum_{n=3}^{\infty} b_n$ converges since it's a p-series with $p = 4 > 1$.

$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{n^4}{n^4 - n^3 - n^2} = 1$;

Since $0 < L < \infty$,

$\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

(c) let $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$;

$\sum_{n=1}^{\infty} b_n$ converges as a geometric series with $|r| = \frac{1}{2} < 1$;

$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{2^n - 1}{2^n} \overset{L'R}{=} \lim_{n\to\infty} \frac{2^n \ln(2)}{2^n \ln(2)} = 1$;

Since $0 < L < \infty$: $\sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

(d) let $a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ and let $b_n = \frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^2\sqrt{n}} = \frac{1}{\sqrt{n}}$;

$\sum_{n=1}^{\infty} b_n$ diverges as a p-series with $p = \frac{1}{2} \leq 1$;

$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^5}} \cdot \frac{\sqrt{5+n^5}}{2n^2+3n}$

$= \lim_{n\to\infty} \left(\frac{n^2}{2n^2+3n}\right) \sqrt{\lim_{n\to\infty} \frac{5+n^5}{n^5}} = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$;

Since $0 < L < \infty$, $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ also diverges by the LCT.

(e) let $a_n = \frac{4^{n+1}}{3^n - 2}$ and $b_n = \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$;

$\sum_{n=1}^{\infty} b_n$ diverges as a geometric series with $|r| = \frac{4}{3} > 1$;

$L = \lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{4^n}{3^n} \cdot \frac{3^n - 2}{4^{n+1}}$

$= \lim_{n\to\infty} \left(\frac{3^n - 2}{3^n}\right) \lim_{n\to\infty} \left(\frac{4^n}{4^{n+1}}\right) = (1)\left(\frac{1}{4}\right) = \frac{1}{4}$;

Since $0 < L < \infty$, $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$ diverges by the LCT;