HW3B. Written Homework 3B. Alternating Series Test. **Due Week 3 Friday 11:59PM**

Name: Answer Key.

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

1

(1) Use the **Alternating Series Test (AST)** to determine the convergence of the following series. If AST can be applied, explicitly show that the conditions on $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$ are satisfied. If AST cannot be applied, state at least one condition that is not satisfied.

Note that there may be other methods to determine the convergence of the following series. However, this problem tests your knowledge and understanding of the Alternating Series Test.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by AST

(b) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges by AST

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{\sin(\pi n+\frac{\pi}{2})}{1+\sqrt{n}}$ converges by AST

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}ne^{-n}$ converges by AST

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{3n-1}{2n+1}$ AST is inconclusive;

(f) $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\cos\left(\frac{\pi}{n}\right)$ AST is inconclusive;

(a) Let $b_n = \frac{1}{n}$;

(1) $b_n > 0$ for all $n \geq 1$;

(2) Method 1: Since $f(x) = \frac{1}{x}$ is decreasing on $[1,\infty)$: $(b_n) = (f(n))$ is also decreasing for $n \geq 1$;

(2) Method 2. For $n \geq 1$: $n+1 > n > 0$;

Then, $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$;

(3) $\lim_{n\to\infty}\frac{1}{n!} = 0$ since $\lim_{n\to\infty} n! = \infty$;

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by AST;

(b) Let $b_n = \frac{n^2}{n^3+1}$ for $n \geq 2$;

(1) For $n \geq 2$, $b_n > 0$;

(2) Let $f(x) = \frac{x^2}{x^3+1}$; Then, $f'(x) = \frac{(x^3+1)(2x)-(x^2)(3x^2)}{(x^3+1)^2}$

$= \frac{x(2x^3+2-3x^3)}{(x^3+1)^2} = \frac{x(-x^3+2)}{(x^3+1)^2}$;

For $x \geq 2$: $x^3 \geq 2$ and $-x^3+2 < 0$;

Since $x > 0$ and $(x^3+1)^2 > 0$,

$f'(x)$ is negative and $f(x)$ is decreasing.

$\therefore (b_n) = (f(n))$ is decreasing;

(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^3+1} = \lim_{n\to\infty}\frac{n^2}{n^3} = 0$;

$\therefore \sum_{n=2}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ converges by AST;

(c) For $n = 2k$ for some $k \in \mathbb{Z}$, i.e. n even:

$\sin(\pi n + \frac{\pi}{2}) = \sin(2\pi k + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1 = (-1)^n$;

For $n = 2k+1$ for some $k \in \mathbb{Z}$, i.e. n odd:

$\sin(\pi n + \frac{\pi}{2}) = \sin(2\pi k + \pi + \frac{\pi}{2}) = \sin(\frac{3\pi}{2}) = -1 = (-1)^n$;

$\therefore \sum_{n=0}^{\infty}\frac{\sin(\pi n+\frac{\pi}{2})}{1+\sqrt{n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$; Let $b_n = \frac{1}{1+\sqrt{n}}$;

(1) For all $n \geq 0$: $b_n > 0$;

(2) Method 1: For $n \geq 0$: $n+1 > n > 0$; $\sqrt{n+1} > \sqrt{n}$;

Then, $b_{n+1} = \frac{1}{1+\sqrt{n+1}} < \frac{1}{1+\sqrt{n}} = b_n$;

(2) Method 2:

Let $f(x) = \frac{1}{1+\sqrt{x}}$; $f'(x) = \frac{-\frac{1}{2}x^{-\frac{1}{2}}}{(1+\sqrt{x})^2} < 0$ for $x \geq 1$;

Since $f(x)$ is decreasing on $[1,\infty)$,

$(b_n) = (f(n))$ is also decreasing for $n \geq 1$;

(3) $\lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$;

$\therefore \sum_{n=0}^{\infty}\frac{\sin(\pi n+\frac{\pi}{2})}{1+\sqrt{n}}$ converges by AST;

(d) Let $b_n = ne^{-n}$; (1) for $n \geq 1$, $b_n > 0$;

(2) let $f(x) = xe^{-x}$; Then, $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(-x+1)$;

for $x > 1$: $-x+1$ is neg. and $f'(x)$ is negative; Then, $f(x)$ is decreasing on $(1,\infty)$;

$\therefore (b_n) = (f(n))$ is decreasing;

(3) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0$;

$\therefore \sum_{n=1}^{\infty} (-1)^n ne^{-n}$ converges by AST;

(e) AST cannot be applied to $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$; let $b_n = \frac{3n-1}{2n+1}$; There are 2 conditions that are not satisfied:

(1) $\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$; AST is inconclusive but the Divergence Test can be used here to conclude divergence;

(2) b_n is increasing for $n \geq 2$: let $f(x) = \frac{3x-1}{2x+1}$; $f'(x) = \frac{(2x+1)(3)-(3x-1)(2)}{(2x+1)^2} = \frac{-3x+5}{(2x+1)^2} > 0$ for $x \geq 2$;

$\therefore (b_n) = (f(n))$ is decreasing for $n \geq 2$;

(f) AST cannot be applied to $\sum_{n=2}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$; let $b_n = \cos\left(\frac{\pi}{n}\right)$ and $b_n > 0$ for $n \geq 2$; There are 2 conditions that are not satisfied;

(1) $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \cos(0) = 1 \neq 0$; AST is inconclusive but the Divergence Test can be used to conclude divergence.

$\boxed{2}$ **(2)** Determine if the following series are either **divergent**, **conditionally convergent**, or **absolutely convergent**. You can use any test as appropriate.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

(b) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$

(c) $\sum_{n=0}^{\infty} \frac{\sin(\pi n + \frac{\pi}{2})}{1+\sqrt{n}}$

(d) $\sum_{n=1}^{\infty} (-1)^n ne^{-n}$

(e) $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

(f) $\sum_{n=2}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$

For the Alternating Series, use results from Problem 1;

(a) let $a_n = \frac{(-1)^{n-1}}{n}$; Then, $|a_n| = \frac{1}{n}$; $\sum_{n=1}^{\infty} |a_n|$ diverges as a p-series with $p = 1 \leq 1$;

Since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent;

(b) let $a_n = \frac{(-1)^{n+1} n^2}{n^3+1}$; Then, $|a_n| = \frac{n^2}{n^3+1}$; let $b_n = \frac{n^2}{n^3} = \frac{1}{n}$; $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as a p-series with $p = 1 \leq 1$;

Since $L = \lim_{n\to\infty} \frac{b_n}{|a_n|} = \lim_{n\to\infty} \frac{n^2}{n^3} \cdot \frac{n^3+1}{n^2} = 1$ and $0 < L < \infty$: $\sum_{n=1}^{\infty} |a_n|$ diverges by LCT;

Since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^3+1}$ is conditionally convergent;

(c) Let $a_n = \frac{\sin(\pi n + \frac{\pi}{2})}{1+\sqrt{n}}$; Then, $|a_n| = \frac{1}{1+\sqrt{n}}$; Let $b_n = \frac{1}{\sqrt{n}}$; Then, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1 \le 1$;

Since $L = \lim_{n\to\infty} \frac{b_n}{|a_n|} = \lim_{n\to\infty} \frac{1+\sqrt{n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$ and $0 < L < \infty$: $\sum_{n=0}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by LCT;

Since $\sum_{n=0}^{\infty} a_n$ converges, $\sum_{n=0}^{\infty} \frac{\sin(\pi n + \frac{\pi}{2})}{1+\sqrt{n}}$ is conditionally convergent;

(d) Let $a_n = (-1)^n n e^{-n}$; Then, $|a_n| = n e^{-n}$;

Method 1: Let $f(x) = xe^{-x}$; Then, $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(-x+1)$ and $f'(x) < 0$ for $x > 1$;

Then, $f(x)$ is positive, continuous, and decreasing on $[2,\infty)$ and the Integral Test can be applied;

$$\lim_{x\to\infty} \int xe^{-x}\,dx = \lim_{x\to\infty} \left[-xe^{-x} + \int e^{-x}\,dx\right] = \lim_{x\to\infty}\left(-xe^{-x} - e^{-x}\right) = \lim_{x\to\infty}\left(\frac{-x-1}{e^x}\right) \overset{L'R}{=} 0;$$

$\left[\begin{array}{ll} u = x; & dv = e^{-x}dx; \\ du = dx; & v = -e^{-x}; \end{array}\right]$ By the Integral Test, $\sum_{n=2}^{\infty} n e^{-n}$ converges;

Since $\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$ is absolutely convergent;

Method 2: Use the Ratio Test, covered after Alternating Series Test;

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = e^{-1} < 1$; By the Ratio Test, $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$ is absolutely convergent;

(e) Since $\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \ne 0$, $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1} \ne 0$; $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ diverges by the Divergence Test;

(f) Since $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, $\lim_{n\to\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right) \ne 0$; $\sum_{n=2}^{\infty} (-1)^{n+1} \cos\left(\frac{\pi}{n}\right)$ diverges by the Divergence Test;

$\boxed{3}$ **(3)** Find an approximation A of $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ accurate to 3 decimal places.

The accuracy claim must be justified using some approximation error theorem.

Strategy: Use the Alternating Series Estimation Theorem (ASET):

Part (1): Show that S converges by AST;

Let $b_n = \frac{1}{n^6}$; Since $b_n > 0$ and $b_{n+1} \le b_n$ for $n \ge 1$ and $\lim_{n\to\infty} \frac{1}{n^6} = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ converges by AST;

$\therefore$ We can apply ASET;

Part (2): By ASET: $|S - S_N| < b_{N+1}$; To find an approximation accurate to 3 decimal places, we want to find $N \in \mathbb{Z}$ minimal s.t. $|S - S_N| < b_{N+1} < \frac{1}{2}(10^{-3})$;

Equivalently, find $x \in \mathbb{R}$ maximal such that $\frac{1}{(x+1)^6} = \frac{1}{2}(10^{-3})$ and let $N = \lceil x \rceil$ since $\frac{1}{x^6}$ is decreasing

$(x+1)^6 = 2(10^3)$; $x + 1 \approx 3.55$; $x \approx 2.55$; Choose $N = \lceil 2.55 \rceil = 3$ and let $A = S_3$;

Then, $|S - A| < \frac{1}{2}(10^{-3})$;

Part (3): $A = S_3 = \sum_{n=1}^{3} \frac{(-1)^{n+1}}{n^6} = 1 - \frac{1}{2^6} + \frac{1}{3^6} \approx \boxed{0.986}$; rounded to 3 decimal places.