HW4A. Written Homework 4A. Root Test and Ratio Test. **Due Week 4 Wednesday 11:59PM**

Name: Answer Key

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

(1) Apply the **Root Test** or the **Ratio Test** on the following series and state the result.

Note that there may be other methods to determine the convergence of the following series. However, this problem tests your knowledge and understanding of the Root Test and the Ratio Test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ **(d)** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^4+n^3+n^2}$ **(g)** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln(n))^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ **(e)** $\displaystyle\sum_{n=2}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ **(h)** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ **(f)** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **(i)** $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

(a) Let $a_n = \frac{1}{n!}$ and use the Ratio Test; $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{n!}{(n+1)!} = \lim\limits_{n\to\infty}\frac{n!}{(n+1)n!} = \lim\limits_{n\to\infty}\frac{1}{n+1} = 0 < 1$;

By the Ratio Test with $L<1$: $\sum\limits_{n=1}^{\infty}\frac{1}{n!}$ converges;

(b) Let $a_n = \frac{(-1)^n n^3}{3^n}$ and use the Ratio Test; $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3} = \left(\frac{1}{3}\right)\lim\limits_{n\to\infty}\frac{(n+1)^3}{n^3} = \frac{1}{3} < 1$;

By the Ratio Test with $L<1$: $\sum\limits_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$ is absolutely convergent;

(c) Let $a_n = \left(\frac{2n+3}{3n+2}\right)^n$ and use the Root Test; $L = \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim\limits_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3} < 1$;

By the Root Test with $L<1$: $\sum\limits_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ is convergent;

(d) Let $a_n = \frac{(-1)^n}{n^4+n^3+n^2}$ and use the Ratio Test; $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{n^4+n^3+n^2}{(n+1)^4+(n+1)^3+(n+1)^2} = \lim\limits_{n\to\infty}\frac{n^4}{(n+1)^4} = 1$;

Since $L=1$, the Ratio Test is inconclusive;

(e) Let $a_n = \frac{10^n}{(n+1)4^{2n+1}} = \frac{10^n}{4(n+1)16^n} = \frac{1}{4(n+1)}\left(\frac{5}{8}\right)^n$; $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{n+1}{4(n+2)}\left(\frac{5}{8}\right)^{n+1}\left(\frac{5}{8}\right)^{-n} = \frac{1}{4}\left(\frac{5}{8}\right) = \frac{5}{32}$;

By the Ratio Test with $L<1$: $\sum\limits_{n=2}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ converges;

(f) Let $a_n = \frac{n^n}{n!}$ and use the Ratio Test;

$$L = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1;$$ By definition, $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$;

$\therefore \sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges by the Ratio Test;

(g) Let $a_n = \frac{(-1)^{n-1}}{(\ln(n))^n}$ and use the Root Test; $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln(n))^n}} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0 < 1$;

By the Ratio Test, $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln(n))^n}$ converges;

(h) Let $a_n = \frac{\sqrt{n}}{1+n^2}$ and use the Ratio Test;

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{n^2+1}{1+(n+1)^2} \sqrt{\lim_{n\to\infty} \frac{n+1}{n}} = 1;$$ The Ratio Test is inconclusive;

(i) Let $a_n = ne^{-n}$;

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = e^{-1} < 1;$$ By the Ratio Test, $\sum_{n=1}^{\infty} ne^{-n}$ is absolutely convergent;