HW4B. Written Homework 4B. Power Series Representations. **Due Week 4 Friday 11:59PM**

Name: Answer Key

Instructions: Upload a pdf of your submission to **Gradescope**. This worksheet is worth 20 points: up to 8 points will be awarded for accuracy of certain parts (to be determined after the due date) and up to 12 points will be awarded for completion of parts not graded by accuracy.

Q1.

(1) For each of the given power series, find the radius of convergence R and the interval of convergence I. You may use any applicable convergence test covered in this course.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x-3)^n$ **(b)** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n+1}x^{2n+1}$ **(c)** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(a) The series can be expressed as a geometric series.

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(x-3)^n = \sum_{n=0}^{\infty} \frac{1}{2}\frac{(x-3)^n}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x-3}{2}\right)$ converges if and only if $\left|\frac{x-3}{2}\right| < 1$;

Then, $|x-3| < 2$; Radius of Convergence: $\boxed{R=2}$;

Interval of Convergence: $|x-3| < 2$; $-2 < x-3 < 2$; $1 < x < 5$; $\boxed{I=(1,5)}$;

Note: The endpoints are already examined using the geometric series test.
If the Ratio Test were used here, different tests have to applied for $x=1$ and $x=5$.

(b) Apply the Ratio Test: $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{2(n+1)+1}\cdot\frac{2n+1}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\cdot\frac{2n+1}{2n+3}\right| = |x^2|\left(\frac{2}{2}\right) = |x^2|$;

For $L = |x^2| < 1$: $|x^2| = |x|^2 < 1$; $|x| < 1$; Then, $\boxed{R=1}$;

For the interval of convergence, test $L = |x|^2 = 1$; $x = \pm 1$:

Assume $x=1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2n+1}(1)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2n+1}$; let $b_n = \dfrac{1}{2n+1}$;

Then, (1) b_n is positive for all $n \geq 0$;

(2) Since $2(n+1)+1 = 2n+3 > 2n+1$, $b_{n+1} < b_n$ for all $n \geq 0$;

(3) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{2n+1} = 0$;

$\therefore$ The power series converges for $x=1$ by the Alternating Series Test;

Assume $x=-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2n+1}(-1)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{2n+1}(-1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ also converges by AST as with $x=1$;

$\therefore \boxed{I=[-1,1]}$;

(c) Apply the Ratio Test: $\displaystyle L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n}\right| = 0$ since x is constant with respect to n;

For all $x \in \mathbb{R}$, $L = 0 < 1$. $\therefore \boxed{R=\infty \text{ and } I=(-\infty,\infty)}$;

Q2. **(2)** Let $f(x) = \sin(x^2)$.

(a) Let $g(x) = \sin(x)$ and let $T_n(x)$ be the n^{th} order Taylor polynomial of $g(x)$ about $x = 0$. Let $R_n(x) = g(x) - T_n(x)$, the n^{th} order remainder of $T_n(x)$ for all $n \geq 0$.

Using Taylor's Inequality, find all $x \in \mathbb{R}$ such that $\lim_{n\to\infty} R_n(x) = 0$.

(b) Find a power series representation for $g(x)$ and identify its radius of convergence R_g.

(c) Determine a power series representation for $f(x)$ and identify its radius of convergence R_f.

(d) Approximate $\int_0^{0.5} \sin(x^2)\,dx$ to 6 decimal places. Justify why your approximation is valid.

(a) By Taylor's Inequality: $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}M$ with $M > 0$ such that $|g^{(n+1)}(x)| \leq M$ on $[0, x]$;

Since $g^{(n)}(x) = \begin{cases} \sin(x) & n = 4k \\ \cos(x) & n = 4k+1 \\ -\sin(x) & n = 4k+2 \\ -\cos(x) & n = 4k+3 \end{cases}$ if for some $k \in \mathbb{Z}$, we can choose $M = 1$ for all $x \in \mathbb{R}$.

Then, $0 \leq |R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$ for all $n \geq 0$; By the Squeeze Theorem, $0 \leq \lim_{n\to\infty} |R_n| \leq \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$;

$\boxed{\therefore \lim_{n\to\infty} R_n(x) = 0 \text{ for all } x \in \mathbb{R};}$

(b) Since $\lim_{n\to\infty} R_n = 0$ for all $x \in \mathbb{R}$, the Taylor series $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$ is a power series representation of $g(x)$ whenever it converges;

Then, $g^{(n)}(0) = \begin{cases} \sin(0) = 0 & n = 4k \\ \cos(0) = 1 & n = 4k+1 \\ -\sin(0) = 0 & n = 4k+2 \\ -\cos(0) = -1 & n = 4k+3 \end{cases}$ if for some $k \in \mathbb{Z}$; Observe that $g^{(n)}(0) = 0$ for n even;

Reindex $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{m=0}^{\infty} \left[\cancel{\frac{g^{(2m)}(0)}{(2m)!} x^{2m}}^{\,0} + \frac{g^{(2m+1)}(0)}{(2m+1)!} x^{2m+1} \right] = \sum_{m=0}^{\infty} \frac{g^{(2m+1)}(0)}{(2m+1)!} x^{2m+1}$;

i.e. skipping the even indices; This is fine regrouping is fine since the partial sums are unchanged.

Observe that if $m = 2k$, i.e. even: $2m+1 = 2(2k)+1 = 4k+1$; Then, $g^{(2m+1)}(0) = 1 = (-1)^m$;

if $m = 2k+1$, i.e. odd: $2m+1 = 2(2k+1)+1 = 4k+2+1 = 4k+3$; Then, $g^{(2m+1)}(0) = (-1) = (-1)^m$;

Finally, $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \boxed{\sum_{m=0}^{\infty} \frac{(-1)^n}{(2m+1)!} x^{2m+1}}$; ← As we should expect if we look this up;

So, $L = \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to\infty} \left| \frac{x^{2m+3}}{(2m+3)!} \cdot \frac{(2m+1)!}{x^{2m+1}} \right| = |x^2| \lim_{m\to\infty} \frac{1}{(2m+3)(2m+2)} = 0$; By the Ratio Test, $\boxed{R_g = \infty}$;

(c) Since $g(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$ on $\mathbb{R}$ and $f(x) = g(x^2)$: $f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} (x^2)^{2m+1} = \boxed{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{4m+2}}$;

By the Ratio Test, $L = \lim_{m\to\infty} \left| \frac{b_{m+1}}{b_m} \right| = 0$ with $b_m = \frac{(-1)^m}{(2m+1)!} x^{4m+2}$ and $\boxed{R_f = \infty}$;

(d) Approximate $I = \int_0^{\frac{1}{2}} \sin(x^2)\,dx$ to 6 decimal places.

Since $f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{4m+2}$ on $\mathbb{R}$ and $[0, \frac{1}{2}] \subseteq \mathbb{R}$: $I = F(\frac{1}{2}) - F(0)$ with

$$F(x) = \int \sin(x^2)\,dx = \int \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{4m+2}\,dx = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \int x^{4m+2}\,dx = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(4m+3)} x^{4m+3};$$

Then, $F(0) = 0$ and $F(\frac{1}{2}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(4m+3)} \left(\frac{1}{2}\right)^{4m+3}$ is an alternating series.

Let $b_m = \frac{(-1)^m}{(2m+1)!\,(4m+3)} \left(\frac{1}{2}\right)^{4m+3}$; Then, b_m is positive; $b_{m+1} < b_m$ for all $m \geq 0$; $\lim_{m \to \infty} b_m = 0$;

We can apply the Alternating Series Approximation Theorem where $|F(\frac{1}{2}) - S_m| < b_{m+1}$ for all $m \geq 0$;

We want to find M such that $|F(\frac{1}{2}) - b_M| < b_{M+1} < \frac{1}{2}(10^{-6}) = 5 \times 10^{-7}$;

By brute force: For $M=0$: $b_1 = 0.000\,187$;
For $M=1$: $b_2 = 3.70 \times 10^{-7}$;
For $M=2$: $b_3 = 4.04 \times 10^{-10}$; This works!

Finally, $I = \int_0^{\frac{1}{2}} \sin(x^2)\,dx \approx S_2 = \sum_{m=0}^{2} \frac{(-1)^m}{(2m+1)!\,(4m+3)} \left(\frac{1}{2}\right)^{4m+3} = \boxed{0.041\,481}$; rounded to 6 decimal places.